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Analytic Vectors, Anomalies and Star Representations

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Abstract. It is hinted that anomalies are not really anomalous since (at least in characteristic examples) they can be related to a lack of common analytic vectors for the Hamiltonian and the observables. We reanalyze the notions of analytic vectors and of local representations of Lie algebras in this light, and show how the notion of preferred observables introduced in the deformation (star product) approach to quantization may help give an anomaly-free formulation to physical problems. Finally, some remarks are made concerning the applicability of these considerations to field theory, especially in two dimensions.

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1. Introduction and Overview

The notion of analytic vectors in Lie group representations goes back to the Fifties, culminating in the works of Nelson [1] and Gårding [2] who proved the density of the space of analytic vectors for representations of Lie groups in Banach spaces. In the Sixties, in connection with the mass-spectrum of elementary particles, some local representations (non integrable to the Lie group) of Lie algebras, and some related infinite-dimensional groups, were considered [3]. This raised the question, developed in the first half of the Seventies [4], of giving criteria for the integrability to the Lie group of Lie algebra representations, based on individual generators (and not on the Lapacian as in Nelson's approach), which also lead to some 'Hartogs-like' theorems on separate and joint analyticity of vectors in Lie group representations: it turns out that, here also, the group structure plays a simplifying role.

Simultaneously with these last developments, since the middle of the Seventies, the deformation (star-product) approach to quantization [5] and to group representations [6] is being developed. In contradistinction to the geometric quantization program, the observables are and remain functions (or distributions) on phase-space, only their composition law reflects quantization when the deformation parameter takes a nonzero value. Such an autonomous approach permits quantizations on nonflat phase spaces, where another Lie algebra, in general, takes the role of the Heisenberg (or of the inhomogeneous symplectic) algebra. In this case, phase-space is generally an orbit of the coadjoint representation of that Lie algebra (of 'preferred observables') and this in turn paves the way both to realizing group representations by means of star-products

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and to a better understanding of the good choice of observables in a given physical situation.

Indeed, the question of quantization anomalies in field theoretical problems (in $1 + 1$ or $3 + 1$ spacetime dimensions) has, during the last few years (although the problem is not new), gained considerable attention and has been related to geometrical notions. In order to understand better the problems involved, some quantum-mechanical anomalies, to which some of the $1 + 1$-dimensional situations can, in a sense, be reduced, have recently been considered (see, e.g., [7]). Our contention in this Letter is that, at least in those relatively simple situations where it is easier to understand what is happening, the anomalies are quite normal. Essentially, they are induced by forcing a canonical type of quantization in a constrained situation where a different one is in order, as exemplified by the star-product formalism. We are then forced to a local representation of the Heisenberg Lie algebra by the lack of common analytic vectors (for the Heisenberg generators and/or for the Hamiltonian and some observables considered). And the anomalies that appear for these observables, anomalies of the Dirac δ type (as in most cases), are no more anomalous than the Heaviside function is with respect to derivation. As a matter of fact, we shall see that by restricting ourselves to a proper set of observables, inspired by the preferred observables occurring in star-product quantization, these anomalies can be avoided.

It is therefore suggested that, in field theoretical models which are fashionable now, the use of star-product quantization may avoid anomalies. And if we insist in considering anomalies, we should at least understand better their origin in terms of lack of common analytic vectors and of the quantization formalism which is used. In turn, his will help, e.g., in understanding just how fundamental is the cancellation of anomalies in 26- or 10-dimensional theories.

The Letter is organized as follows: in Section 2, we briefly review some results on the theory of analytic vectors and show how what had been called 'local representations' can be understood as 'anomalous representations' in today's terminology. We also tackle the associated infinite-dimensional groups, in view of their relationship with the notion of preferred observables in star quantization. In Section 3, we treat in this light some simple examples of anomalies in quantum mechanics and indicate their relevance to field theory. In Section 4, we show how star-product quantization, and more specifically in the above examples, the replacement of the Heisenberg algebra by the Euclidean $E(2)$ algebra, may help us avoid anomalous 'observables'. The relevant star-product for $E(2)$ is exhibited. Finally, in Section 5, we consider the applicability of these considerations to simple field-theoretical models.

2. Analytic Vectors and Anomalous Representations

2.1. We recall that a vector φ in a Banach (or quasi-complete locally convex) space H on which is defined a (continuous) representation of a (finite-dimensional) Lie group G, is said analytic (resp. differentiable) if the (continuous) map $G \ni g \to U(g)\varphi \in H$ is analytic (resp. differentiable). According to a result of Gårding completed by Dixmier