Quaternionic, Quaternionic Kähler, and Hyper-Kähler Supermanifolds

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Abstract. Almost quaternionic, quaternionic, hyper-Kähler, and quaternionic Kähler supermanifolds are introduced and studied.


1. Introduction

According to the classification of Riemannian manifolds by their holonomy [1, 2], the holonomy group of an oriented Riemannian manifold which is neither locally symmetric nor irreducible, must occur in the following list: SO(n), U(m), SU(m), Sp(k), Sp(k)Sp(1). The last two groups, namely G_2 and Spin(7), occur only in dimensions 7 and 8, respectively. The group SO(n) corresponds to the 'generic' geometry, the groups U(m), SU(m), and Sp(k) correspond to Kähler manifolds of various degree of speciality, and the remaining group Sp(k)Sp(1) = S(k) × Sp(1)/Z_2 corresponds to the family of quaternionic Kähler manifolds of dimension 4k, k ≥ 2.

Quaternionic Kähler and hyper-Kähler manifolds are of increasing interest to both mathematicians and physicists (see, e.g., [3–11]). One of the reasons for this is that the classical σ-model, which consists of the energy functional for maps f: Σ → X of a compact Riemannian 4-manifold into a Riemannian manifold X, admits locally an N = 2 supersymmetric extension only when X is a quaternionic Kähler manifold with negative scalar curvature [4]. In other words, hyper-Kähler and quaternionic Kähler geometries determine N = 2 locally supersymmetric interactions of bosons and fermions in four-dimensional spacetime.

The purpose of this Letter is to construct proper supersymmetric extensions of hyper-Kähler and quaternionic Kähler manifolds. The starting point is the definition of an almost quaternionic supermanifold which is given in Section 2. Salamon [5], in his definition of an almost quaternionic manifold, used the G-structure (with G ≅ GL(k, C)Sp(1)) on the quaternionic projective space HP^k as a model. In a full analogy with Salamon's approach, one might note that tangent bundle of the
simplest quaternionic projective superspace $\mathbb{HP}^k|l$ (which may be defined as the quotient $(\mathbb{H}^{k+1} \oplus \mathbb{H}^{l+1})/\mathbb{H}^*$, where $\mathbb{H}^*$ is the group of non-zero quaternions acting by right multiplication and $\Pi$ denotes the parity change functor) admits a $G$-structure with $G \cong GL(k \mid 1, \mathbb{H})Sp(1)$ and define an almost quaternionic superspace as a real $(4k \mid 4)$-dimensional supermanifold whose tangent bundle admits $GL(k \mid 1, \mathbb{H})Sp(1)$-structure. However, this naive suggestion seems to be unsuccessful, because it is closely related to the straightforward supersymmetric generalization of the Riemannian geometry using the notion of supermetric (it is well known that a supermetric version of super-Riemannian geometry proved to be inadequate because of the redundantly rich supermultiplet structure of a supermetric). Motivated by this fact we choose an alternative way to the definition of an almost quaternionic supermanifold which provides a natural generalization of Ogievetsky–Sokatchev–Manin approach to $N = 1, D = 4$ supergravity [12, 13].

The Letter is organized as follows. In Section 2, an almost quaternionic supermanifold of dimension $4k \mid 2k + 2, k \geq 2$, is defined and its local structure is investigated. In Section 3, we introduce the notion of a quaternionic supermanifold $\mathcal{M}$ and give explicit expressions of torsion and curvature tensors of a canonical super-connection on $\mathcal{M}$. The latter are used to prove the following two statements: (i) the underlying $4k$-dimensional manifold of a quaternionic supermanifold has the induced quaternionic structure; (ii) to any $(4k \mid 2k + 2)$-dimensional quaternionic supermanifold $\mathcal{M}$ there corresponds a $(2k + 1 \mid 1)$-dimensional twistor superspace parametrizing $\alpha$-supersurfaces in $\mathcal{M}$ (cf. [3, 9, 10]). The final Section 4 is devoted to proper supersymmetric generalizations of categories of hyper-Kähler and quaternionic Kähler manifolds.

### 2. Almost Quaternionic Supermanifolds

Following [13], we first make a brief review on classification of various real structures on a complex supermanifold. Let $\mathcal{M}$ be a $(m \mid n)$-dimensional complex supermanifold, i.e. a pair, $(\mathcal{M}_m, \mathcal{O})$, consisting of an ordinary $m$-dimensional complex manifold $\mathcal{M}_n$, together with a structure sheaf $\mathcal{O}$ of supercommutative superalgebras over $\mathbb{C}$ satisfying the condition that locally $\mathcal{O}|' \cong \mathcal{O}(U) \otimes \Lambda^\alpha V$ for some vector space $V$ of dimension $n$, where $\mathcal{O}(U)$ denotes the algebra of holomorphic functions on a domain $U \subset \mathcal{M}_n$. A real structure on $\mathcal{M}$ of the type $(e_1, e_2, e_3)$, where $e_i = \pm 1, i = 1, 2, 3$, is an even $\mathbb{R}$-linear mapping $\rho : \mathcal{O} \to \mathcal{O}$, $a \to a^\rho$, with the following properties

$$(za)^\rho = \bar{z}a^\rho, \quad a^\rho = (e_i)^\rho a, \quad (ab)^\rho = e_3(e_2)^\rho b^\rho a^\rho,$$

where $\bar{a}$ denotes the parity of a function $a \in \mathcal{O}$.

If $\mathcal{V}$ is a vector bundle on $\mathcal{M}$, then a prolongation $\hat{\rho}$ of type $\eta = \pm 1$ of a given real structure $\rho : \mathcal{O} \to \mathcal{O}$ on $\mathcal{M}$ is an even $\mathbb{R}$-linear mapping $\hat{\rho} : \mathcal{V} \to \mathcal{V}$, $v \to v^\rho$, with the properties

$$v^\rho = \eta(e_i)^\rho v, \quad (av)^\rho = e_3(e_2)^\rho v^\rho a^\rho, \quad (va)^\rho = e_3(e_2)^\rho v^\rho a^\rho.$$