EQUATIONS WITH SOLITON SOLUTIONS AND THE PSEUDOSPHERE

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1. INTRODUCTION

Sasaki [1, 2] and Crampin [3] are two of a number of authors who discuss soliton solutions of some nonlinear partial differential equations from a geometrical viewpoint. They relate the Backlund transformations which can be used to generate n-soliton solutions of such equations as the two-dimensional Sine–Gordon and Korteweg–de Vries (KdV) equations to mappings of one two-dimensional pseudospherical surface into another. The SL(2, \( \mathbb{R} \)) symmetry admitted by these pseudospherical surfaces are related to the SL(2, \( \mathbb{R} \)) symmetry admitted by these differential equations. Wahlquist and Estabrook [4, 5] make a different geometric approach in discussing prolongation structures of these non-linear equations. Sasaki and Crampin tie up some aspects of their discussion of pseudospherical surfaces with the prolongation variables and their defining equations as given by Wahlquist and Estabrook.

In this paper it is shown that there are even closer links between these approaches than are apparent in [1, 3]. Three different defining equations for three prolongation variables \( z_4, z_6 \) and \( z_8 \) (or, equivalently, \( z_5, z_7 \) and \( z_8 \)), as discussed in [2], are shown to be related to finding canonical variables for the pseudosphere. These canonical variables are \( z_6 \) and \( z_8 \). The KdV equation is discussed in order to highlight these links and to show how these prolongation variables for this equation, as discussed in [4], which are related to SL(2, \( \mathbb{R} \)) symmetry, arise naturally from the viewpoint of this paper.

It is shown that the isometries of the pseudosphere give rise to another pseudosphere and that this fits into the structure discussed in the first part of the paper. The last section shows directly how a connection, which comes from flat space, is associated with the structure equations of the pseudosphere.

2. BASIC STRUCTURE

Sasaki [1] develops his theory on the basis of the vanishing of a two-form

\[ \Theta = d\Omega - \Omega \wedge \Omega = 0 \]  \hspace{1cm} (2.1)

where

\[ \Omega \equiv d\Omega - \Omega \wedge \Omega = 0 \]
\[ \Omega = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & -\omega^1 \end{pmatrix} \]  \hspace{1cm} (2.2)

and where the zero two-form (2.1) gives the original non-linear partial differential equation to be solved. Equations (2.1) and (2.2) can be written as the Maurer–Cartan equations

\[ d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = 2\omega^1 \wedge \omega^3, \quad d\omega^3 = -2\omega^1 \wedge \omega^2. \]  \hspace{1cm} (2.3)

The association

\[ \Omega = \begin{pmatrix} -\frac{1}{2} \sigma^2 & \frac{1}{2} (\omega + \sigma^1) \\ \frac{1}{2} (\omega + \sigma)^1 & -\frac{1}{2} \sigma^2 \end{pmatrix} \]  \hspace{1cm} (2.4)

or a gauge-equivalent association, in (2.3) and (2.4) gives the structure equations

\[ d\sigma^1 = \omega \wedge \sigma^2, \quad d\sigma^2 = -\omega \wedge \sigma^1, \quad d\omega = \sigma^1 \wedge \sigma^2 \]  \hspace{1cm} (2.5)

for the pseudosphere with metric

\[ ds^2 = \sigma^1^2 + \sigma^2^2 \]  \hspace{1cm} (2.6)

and with constant negative Gaussian curvature \( K = -1 \).

\( \Omega \) here appears in a pair of completely integrable Pfaffian equations

\[ \dot{\nu} = \Omega \nu, \quad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \]  \hspace{1cm} (2.7)

which arises in the inverse scattering method often discussed for these differential equations. The Equations (2.1) and (2.7) are invariant under the 'gauge' transformation

\[ \nu \rightarrow \nu' = A\nu \quad \Omega \rightarrow \Omega' = dAA^{-1} + A\Omega A^{-1} \]  \hspace{1cm} (2.8)

where \( A \) is an arbitrary \( 2 \times 2 \) matrix with unit determinant, \( \det A = 1 \).

3. USE OF THE GAUGE TRANSFORMATION

For any form of \( \Omega \) in (2.2), the gauge \( A \) can also be chosen so that, from (2.8),

\[ \omega^2' = 0 \]  \hspace{1cm} (3.1)