On the Riemann Theta Function of a Trigonal Curve and Solutions of the Boussinesq and KP Equations

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Abstract. Recently, considerable progress has been made in understanding the nature of the algebro-geometrical superposition principles for the solutions of nonlinear completely integrable evolution equations, and mainly for the equations related to hyperelliptic Riemann surfaces. Here we find such a superposition formula for particular real solutions of the KP and Boussinesq equations related to the nonhyperelliptic curve \( \omega^4 = (\lambda - E_1)(\lambda - E_2)(\lambda - E_3)(\lambda - E_4) \). It is shown that the associated Riemann theta function may be decomposed into a sum containing two terms, each term being the product of three one-dimensional theta functions. The space and time variables of the KP and Boussinesq equations enter into the arguments of these one-dimensional theta functions in a linear way.

1. Introduction

Until recent times, simple examples of nonhyperelliptical solutions of the Boussinesq equation

\[ 3u_{xy} + (u_{xxx} + 6uu_x)_x = 0, \quad (1) \]

reducible to one-dimensional theta functions, were unknown. The curve \( \Gamma \):

\[ \omega^4 = (\lambda - E_1)(\lambda - E_2)(\lambda - E_3)(\lambda - E_4), \quad \text{Im} E_k = 0, \]

was discussed in [1] as an example of Krichever's reduction of the KP equation to a Boussinesq equation via Weierstrass points. But at that time only the Weierstrass points coinciding with branch points were explored, but the possibility of reducing associated three-dimensional theta functions to one-dimensional theta functions was not discussed. Such a possibility arises from the existence of the conformal automorphism \( \tau : (\omega, \lambda) \rightarrow (i\omega, \lambda) \) interchanging the sheets of the associated Riemann surface, realized as a four-sheeted covering of the complex \( \lambda \)-plane. The way to explore this automorphism for reducing three-dimensional Riemann theta functions goes back to the

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methods given in [2, 3] — see also [4] — although application of the matrix version of Appel’s theorem used below simplifies the calculations. As a main result we obtain a family of the genus three solutions of the KP and Boussinesq equations expressed by means of elliptic theta functions.

2. The Algebrogeometrical Solutions of the KP and Boussinesq Equations

The formula

\[ u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \ln \theta(xU + yV + tW - I \mid B) + C \]  

(2)

describes the solutions of the Kadomtsev–Petviashvili (KP) equation

\[ 3u_{yy} + (u_{xx} + 6uu_x - 4u_t) = 0, \]  

(3)
generated by an arbitrary compact Riemann surface \( \Gamma \) [5]. In Equation (2) \( B \) means the matrix of \( b \) periods of the surface \( \Gamma \) in some canonical basis of \( H_1(\Gamma) \). \( \theta \) is a \( g \)-dimensional theta function defined by the formula

\[ \theta(p \mid B) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (p \mid B), \]  

(4)

\[ \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (p, B) = \sum_{m \in \mathbb{Z}^g} \exp \{ \pi i \langle B(m + \alpha, m + \beta) + 2\pi i \langle m + \alpha, p + \beta \rangle \} , \]  

(5)

where \( g \) is the genus of the Riemann surface \( \Gamma \). \( U, V, W \) are the vectors of \( b \)-periods of some normalized Abelian integrals of the second kind with the poles at the marked point \( P_0 \in \Gamma \). In the case of the hyperelliptic curve, \( P_0 \) may coincide with one of the branch points. In this case, vector \( V \) turns out to be equal to zero and formula (2) reduces to the solution of the KdV equation, given by the Its–Matveev formula [1, 6–8]. If \( \Gamma \) is some trigonal curve, i.e., there exists a meromorphic function with a unique pole at the point \( P_0 \) of the order 3, it turns out that \( W = 0 \). In this case, the KP solution (2) is independent on \( t \) and satisfies the nonlinear Boussinesq equation (1). Trigonal curve 3 of the genus \( g = 3 \) cannot be hyperelliptic. The curve \( \Gamma \) considered in this Letter is nonhyperelliptic of the genus 3. It is of the nondividing type. Its branch points are \( P_j = (0, E_j) \).

Let \( P_0 \) coincide with one of the branch points. It is possible to construct a local parameter \( K(P) \), \( P \in \Gamma \) in such a way that under the action of the antiholomorphic involution \( \tau : (\omega, \lambda) \rightarrow (\overline{\omega}, \overline{\lambda}) \), it transforms to \( \overline{\kappa} \) or \( -\overline{\kappa} \). Let \( \gamma(y) \) be the path with the starting point at \((0, E_j)\) and ending at \((0, E_j)\), \( \arg \omega(\lambda) = (\pi m)/4 \) along the path. The canonical basis of the curve \( \Gamma \) may now be defined by

\[ a_1 = \gamma_{34}(1) + \gamma_{31}(-1), \quad a_2 = \gamma_{34}(-3) + \gamma_{33}(3), \]