GEOMETRY OF GAUGE FIELDS IN A MULTIDIMENSIONAL UNIVERSE*

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ABSTRACT. Let S be a group of automorphisms of a principal fibre bundle (U, π, E, R), both
groups S and R being compact. Let I (resp. H) be the isotropy group of S (resp. S × R) acting on
E (resp. U), and let N(I) (resp. N(H)) be the normalizer of I (resp. H) in S (resp. S × R). We con-
struct two principal bundles P(M, N(I) \mid I) \subset E
and
Q(M, N(H) \mid H) \subset U, where M = E/S is the
space of orbits of S in E, and we prove that, given a connection A in P, there is a one-to-one
correspondence between S-invariant connections ω in U and triples (B, ϕ, ψ), where B is a
connection in Q, part of which is a pullback π*A of A, and ϕ, ψ are scalars which are cross-
sections of certain vector bundles associated with Q. The resulting final gauge group
N(H) \mid H
is
shown to contain as a normal subgroup the 'centralizer of I in R', known from earlier works of
other authors. A dimensional reduction of the Einstein—Yang—Mills system on E is briefly discussed.

1. INTRODUCTION

In this note we study gauge fields in a higher-dimensional spacetime (multidimensional universe) E.
These gauge fields are constrained to be invariant under a given global group S of transformations
of E. A particular case of S acting transitively on E is analyzed in a mathematical literature (Wang's
Theorem [1]), but it is too restrictive for physical applications. Indeed, we tend to believe that our
four-dimensional space-time M is precisely just the space of orbits of some S on E. Generalizations
of the Wang theorem were studied by many authors in the context of symmetric monopole
solutions and dimensional reduction [2–7], and many models were constructed [8–10]. Both
local (in terms of Lie derivatives) [2], and global (involving finite group transformations) [3–5]
formulations of invariance of gauge fields were given. However, it was always assumed that the
structure of E is that of the product E = M × S/I of M and of the typical orbit S/I, I being the

*On leave from the Institute of Theoretical Physics, University of Wroclaw, Wroclaw, Poland.
**Partially supported by the Polish Ministry of Science, Higher Education and Technology under the Project MRI-7.
isotropy group. As it is shown in a recent study by Coquereaux and Jadczyk [11], the most general \( S \)-invariant metric on \( E \) is not just a product of metrics on \( M \) and \( S \), but it also involves a connection in the bundle \( P(M, N(I) \mid I) \) (see Section 2.3) which is necessary to 'glue' the two metrics together. In the present paper, we give a complete description of gauge fields invariant under a given symmetry group in full generality, i.e., without the simplifying assumption of the product structure of \( E \).

In Section 2 we analyze an action of a given global symmetry group \( S \) on a principal bundle \( U(E, R) \), and introduce two important principal bundles \( Q(M, N(H) \mid H) \) and \( P(M, N(I) \mid I) \). The effective final gauge group is shown to be \( N(H) \mid H \) where \( N(H) \) is the normalizer of the isotropy subgroup \( H = S \times \sim \). This final gauge group contains, as a normal subgroup, the gauge group (the 'centralizer') of the authors [2-7]. In Section 3, we describe \( S \)-invariant connections in terms of geometrical objects based on \( M \). It is shown that an invariant connection \( \omega \) in \( U \) gives rise to two kinds of scalar fields \((\Phi, \psi)\) and a connection \( B \) in \( Q \), provided a connection \( A \) in \( P \) is given. In the process of dimensional reduction, a natural connection in \( P \) is provided by an \( S \)-invariant Riemannian metric on \( E \), which is necessary for constructing an invariant action functional for \( \omega \). Comments are given in Section 4.

2. GROUP ACTION ON A PRINCIPAL BUNDLE

2.1. Let \( U(E, R) \) be a principal fiber bundle with the base \( E \) and the structure group \( R \), and let there be given a group \( S \) of bundle automorphisms acting on \( U \) from the left. We have

\[(su)r = s(ur), \quad s \in S, \; r \in R, \; u \in U,
\]

where \( su = L_s(u) \) and \( ur = R_r(u) \) denote the images of \( u \) under \( s \in S \) and \( r \in R \) respectively. We assume that \( R \) and \( S \) are compact Lie groups, their actions on \( U \) being smooth. Since the actions of \( R \) and \( S \) commute, the formula

\[\begin{align*}
(s, r) : u &\mapsto R_{(s, r)}(u) = s^{-1}ur
\end{align*}\]

defines the right action of the direct product group \( G \is S \times R \) on \( U \). We assume that this action is regular, i.e., that all the isotropy subgroups

\[H_u = \{(s, r) \in G : su = ur\}, \quad u \in U,
\]

are mutually conjugated in \( G \). Since \( S \) is fiber preserving, its action on \( U \) projects down onto \( E \): \( s\pi(u) = \pi(su) \), where \( \pi : U \to E \) is the bundle projection. The isotropy subgroups of the induced action of \( S \) on \( E \) are denoted by \( I_y, y \in E \). We shall assume that \( I_y \) are connected.

2.2. The following Lemma characterizes \( H_u \):

**Lemma.** Given \( u \in U \) there exists a unique group homomorphism \( \lambda_u : I_{\pi(u)} \to R \) such that