ASYMPTOTIC FIELDS FOR THE QUANTUM NONLINEAR SCHRODINGER EQUATION WITH ATTRACTIVE COUPLING

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ABSTRACT. The quantum nonlinear Schrödinger equation with attractive coupling is considered through the quantum inverse scattering method. The asymptotic fields arising in this model are characterized in terms of scattering data operators.

1. INTRODUCTION

In recent years a great deal of attention has been paid to the analysis of the quantum nonlinear Schrödinger equation by means of the quantum inverse scattering method (QISM) [1–7]. The corresponding Hamiltonian is given by

\[ H = \int dx (\psi_x^+ \psi_x + c \psi_x^+ \psi_x \psi \psi), \]  

(1.1)

where \( \psi(x) \) denotes the one-dimensional boson field satisfying the canonical commutation relations

\[ [\psi(x), \psi(y)] = 0, \quad [\psi(x), \psi(y)] = \delta(x - y). \]  

(1.2)

This model describes a field of bosons of mass \( \frac{1}{2} \) in one space dimension interacting through the two-body potential \( c \delta(x - y) \). For the case of attractive coupling (\( c < 0 \)) the scattering states present groupings of the fundamental bosons into stable fragments due to the presence of bound states. Indeed, for all \( n > 1 \) the restriction of the Hamiltonian to the \( n \)-particle subspace has, after removing the center of mass motion, exactly one bound state. Groupings of \( n \) fundamental bosons behave asymptotically as free particles of mass \( n/2 \). In this paper we are concerned with the asymptotic fields associated with these objects. Our analysis is based on the set of scattering data operators proposed by Göckeler [6] and on the method used by Grosse [8] for the repulsive case. We show that the asymptotic fields admit a simple characterization in terms of the reflection coefficient operators.
2. SCATTERING DATA OPERATORS

The QISM for the nonlinear Schrödinger model is based on a normal ordered operator version of the Zakharov–Shabat spectral problem [9]. We shall adopt the quantum spectral problem used by Göckeler for the attractive case [6]

\[
\frac{\partial}{\partial x} \varphi_1 = -ik \varphi_1 + \sqrt{-c} \varphi_2 \psi(x), \quad \frac{\partial}{\partial x} \varphi_2 = ik \varphi_2 - \sqrt{-c} \psi^\dagger(x) \varphi_1, \quad (2.1)
\]

where \( k \) is the eigenvalue parameter and \( \psi, \psi^\dagger \) and \( \varphi_i (i = 1, 2) \) are considered as operators. In order to define the scattering data operators we consider the solution \( \varphi(k, x) \) of (2.1) verifying

\[
\lim_{x \to -\infty} \exp(ikx\sigma_3)\varphi(k, x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2.2)
\]

For \( \text{Im} \ k > 0 \) the two components of \( \varphi \) may be expressed as formal series of Wick monomials of the fields \( \psi \) and \( \psi^\dagger \) with bounded kernels. By denoting \( \varphi(k, x) = \exp(ikx\sigma_3)\varphi(k, x) \), then for each \( n = 1, 2, \ldots \) there are two scattering data operators given by

\[
B_n^\dagger(-2k) = \lim_{x \to -\infty} \left[ \exp\left(\frac{C}{2}(n-1)K\right) \prod_{j=1}^{n} \hat{\varphi}_2 \left( k - \frac{i}{2}c(n-j), x \right) \exp\left(\frac{C}{2}(1-n)K\right) \right], \quad (2.3a)
\]

\[
A_n(k) = \lim_{x \to -\infty} \left[ \exp\left(\frac{C}{2}(n-1)K\right) \prod_{j=1}^{n} \hat{\varphi}_1 \left( k - \frac{i}{2}c(n-j), x \right) \exp\left(\frac{C}{2}(1-n)K\right) \right], \quad (2.3b)
\]

where \( K \) is the boost operator

\[
K = \int_{-\infty}^{\infty} dx \psi^\dagger(x)x\psi(x). \quad (2.4)
\]

Now we define the reflection coefficient operators by

\[
R_n^\dagger(p) = \frac{1}{n\sqrt{-2nc}}B_n^\dagger \left( \frac{p}{n} \right) \left( A_n \left( -\frac{p}{2n} \right) \right)^{-1}, \quad n = 1, 2, \ldots \quad (2.5)
\]

These expressions are slightly different from the ones used by Göckeler who defines

\[
R_n^\dagger(p) = B_n^\dagger(p)\left( A_n \left( -\frac{p}{2} \right) \right)^{-1}. \quad (2.5)
\]

The reflection coefficient operators (2.5) satisfy the following simple algebraic relations

\[
R_n^\dagger(p)R_n^\dagger(p') = S_{nm} \left( \frac{p}{n} - \frac{p'}{n'} \right) R_n^\dagger(p')R_m^\dagger(p), \quad (2.6a)
\]