NONLINEAR REPRESENTATIONS OF CONNECTED NILPOTENT REAL LIE GROUPS

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ABSTRACT. We consider nonlinear representations such that the linear part is the finite sum of irreducible unitary representations, in which case linearizations and canonical forms (normalizations) are given and we prove the analyticity of these normalizations in good cases.

0. NOTATIONS AND DEFINITIONS

The nonlinear representations theory was studied by Flato et al. [1], where they also studied formal representations, analytic representations and analogy using that theory. We recall here only some elementary notions and definitions.

Let $E$ and $F$ be two Hilbert spaces, we denote by $\mathcal{L}^n(E, F)$ the space of continuous symmetric $n$-linear mappings from $E$ into $F$, and we denote by $\mathcal{F}_0(E, F)$ the set of formal power series of the $\Sigma_{n \geq 1} f^n$ type, where $f^n \in \mathcal{L}^n(E, F)$. We keep the same notation for an element in $\mathcal{L}^n(E, F)$ and its canonical identification with an element of $\mathcal{L}(\hat{\otimes}_n, E, F)$, where $\hat{\otimes}_n, E$ is the symmetrical projective tensor product of $E$ by itself $n$-times.

The product of power series is defined as following: if $f = \Sigma_{n \geq 1} f^n \in \mathcal{F}_0(E, G)$ and $g = \Sigma_{n \geq 1} g^n \in \mathcal{F}_0(F, E)$ then $h = f \circ g = \Sigma_{n \geq 1} h^n \in \mathcal{F}_0(F, G)$ and

$$h_n = \sum_{p=1}^{n} f^p \sum_{i_1 + \ldots + i_p = n} (g^{i_1} \otimes \ldots \otimes g^{i_p}) \circ o_n,$$

where $o_n$ is the symmetrization operator. In particular, if $E = F = G$, then the product of the power series is a composition law in $\mathcal{F}_0(E) = \mathcal{F}_0(E, E)$ and $f = \Sigma_{n \geq 1} f^n$ is invertible in $\mathcal{F}_0(E)$ if and only if $f$ is an automorphism of $E$.

DEFINITION 1. A formal representation of a real Lie group $G$ in $E$ is a morphism $S$ from $G$ to the group of the invertible elements for the composition law in $\mathcal{F}_0(F)$ such that if $S_g = \Sigma_{n \geq 1} S^n_g$, and $x_i \in E$ ($i = 1, \ldots, n$), the mappings $g \rightarrow S^n_g(x_1, \ldots, x_n)$ are continuous.

Now, we define a mapping: $d: \mathcal{F}_0(E) \rightarrow \mathcal{F}(E, L(E))$ by
\[
df^n(x_1, \ldots, x_{n-1})(h) = f^n(h, x_1, \ldots, x_{n-1}) + f^n(x_1, h, x_2, \ldots, x_{n-1}) + \ldots
\]

\[+ f^n(x_1, \ldots, x_{n-1}, h) \quad \text{and} \quad df = \sum_{n>1} df^n.
\]

Set \( \mathcal{A}_0(E) = \mathcal{F}_0(E, \mathbb{C}) \), to \( Y \in \mathcal{F}_0(E) \) we assign \( \widetilde{Y} \) which acts on \( \mathcal{A}_0(E) \) by the formal development of formula \( \tilde{Y} \). \( f(x) = -df_x(Yx) \). Then if we define the brackets in \( \mathcal{F}_0(E) \) by \( [X, Y] = dX(Y(.)) - dY(X(.)) \) we obtain \( \widetilde{[X, Y]} = \widetilde{X} \circ \widetilde{Y} - \widetilde{Y} \circ \widetilde{X} \) [2].

From now on, we identify \( Y \) and \( \widetilde{Y} \). We shall identify \( \mathcal{A}_0(E) \) as the set of formal power series with complex values which vanish at zero, and we identify \( \mathcal{F}_0(E) \) as the set of formal vector fields vanishing at zero. In fact, if we set, for every \( f \) in \( \mathcal{A}_0(E) \), \( \partial f = \delta f/\partial x_i = -df_{(i)}(e_i) \) where \( (e_i)_{i \in I} \) is a Hilbert basis of \( E \), then we prove that \( Y \) is of the type \( \Sigma Y_j \partial_{j} \), with \( Y_j \in \mathcal{A}_0(E) \) and \( (\Sigma Y_j \partial_{j})^n = \lim_{j \to \infty} (\Sigma Y_j \partial_{j})^n \).

Let \( S \) be a formal representation of a Lie group \( G \) in a Hilbert space \( H \) such that \( R_g^n = S_{g-1}^n S_g^n \) is \( C^\infty \) from \( G \) to \( L_n(E) \), then for every \( X \in X \), the Lie algebra of \( G \), the operator

\[
dS_X = dS_X + \sum_{n>2} dS_X^n, \quad \text{where} \quad dS_X^n = \frac{d}{dt} R^n_{\exp(tX)}|_{t=0},
\]

defines a formal representation of \( g \) in the space \( E^\infty \) of \( C^\infty \) vectors for \( S^1 \) ([1], prop. 7). Moreover \( dS_X^n \in L^n(E^\infty) \cap L^\infty(E) \quad \forall n \geq 2\). Conversely if \( S^1 \) is an unitary representation of \( G \) in \( E \) and \( E^\infty \) its set of \( C^\infty \) vectors, and if \( dS_X^n \) are elements of \( L^n(E^\infty) \cap L^\infty(E) \) such that:

\[
dS_X = dS_X^1 + \sum_{n>2} dS_X^n
\]

is a representation of \( g \) in \( E^\infty \), then there exists one and only one representation \( S \) of \( \tilde{G} \), universal covering of \( G \), in \( E \) such that:

\[
dS_X^n = \frac{d}{dt} S^n_{\exp(tX)}|_{t=0} \quad \forall n \geq 2 \quad ([1], \text{prop. 9}).
\]

1. normalization in finitely-dimensional spaces

Let \( G \) be a real connected nilpotent Lie group and let \( E \) be a \( m \)-dimensional Hilbert space. Let \( S \) be a formal representation of \( G \) in \( E \). We suppose that the linear part \( S^1 \) of \( S \) is a finite sum of the unitary characters, \( S^1 = \sum_{j=1}^{m} \chi_j \). If \( g = \exp(X) \), then

\[
dS_X^1 = i \sum_{j=1}^{m} \mu_j(x) x_j \partial_{j}, \quad (x_j)_{j=1}^{n} \quad \text{is a coordinate system. The eigenvalues of the operator} \quad [dS_X^1, \cdot] \quad \text{are of the type}
\]