A NEW METHOD FOR OBTAINING EXACT ANALYTICAL FORMULAE FOR THE ROOTS OF TRANSCENDENTAL FUNCTIONS

E.G. ANASTASSELOU
Division of Applied Mechanics. The National Technical University of Athens, P.O. Box 61028, GR-151.10 Amaroussion, Greece

and

N.I. IOAKIMIDIS
Division of Applied Mathematics and Mechanics, School of Engineering, University of Patras, P.O. Box 1120, GR-261.10 Patras, Greece

ABSTRACT. A new method is proposed for the derivation of closed-form formulae for the zeros and poles of sectionally analytic functions in the complex plane. This method makes use of the solution of the simple discontinuity problem in the theory of analytic functions and requires the evaluation of real integrals only (for functions with discontinuity intervals along the real axis). Many transcendental equations of mathematical physics can be successfully solved by the present approach. An application to such an equation, the molecular field equation in the theory of ferromagnetism, is made and the corresponding analytical formulae are reported together with numerical results.

1. INTRODUCTION

Several problems of mathematical physics and other branches of science reduce to the solution of transcendental equations. In spite of the fact that such an equation can be easily solved by classical numerical methods, the derivation of closed-form formulae for its roots is a challenging possibility for two reasons: first, such a formula is only useful in theoretical investigations; second, this formula is valid if the values of the parameters of the transcendental equation vary, whereas the numerical results are useful only for particular values of the parameters and not of general validity.

An interesting method for obtaining exact analytical formulae for the zeros of sectionally-analytic functions was suggested by Burniston and Siewert [1, 2]. This method has been successfully used in a series of papers [3–12] for the solution of transcendental equations appearing in problems of mathematical physics. The advantage of Burniston and Siewert’s method over the earlier methods for solving exactly transcendental equations due to Delves and Lyness [13] and to Abd-Elall et al. [14] is that this particular method gives the zeros of an analytic function through...
formulae containing only real integrals (for sectionally-analytic functions with a discontinuity interval along the real axis) and not complex contour integrals, as is the case in the methods of [13] and [14]. Of course, in Burniston and Siewert’s method, if the original transcendental function does not possess a discontinuity interval along the real axis, it should be transformed to such an equation; this is generally an easy task.

The disadvantage of Burniston and Siewert’s method is that it makes use of the theory of the Riemann–Hilbert boundary-value problem [15] in the theory of analytic functions, which is sufficiently complicated. The user of this method should be familiar with the concepts of the index and the canonical function of the Riemann–Hilbert problem and with the theory of this problem before applying the above method to a particular equation. This difficulty is avoided in the method proposed here, where a simple discontinuity problem is solved instead of a Riemann–Hilbert problem. Moreover, the present approach requires less computations for the derivation of numerical results by using an $n$-point numerical integration rule for the evaluation of the integrals and the functions $\exp$ and $\tan^{-1}$ (which appear in the formulae in Burniston and Siewert’s method) do not appear in the formulae of the present method. In spite of these differences, both Burniston and Siewert’s method and the present one lead to formulae for the same zeros of an analytic function and these formulae are always equivalent.

2. THE PROPOSED METHOD

We consider a sectionally-analytic function $F(z)$ in the complex plane with an arc of discontinuity $L$. In general, $L$ is a part $[a, b]$ or a union of parts $[a_i, b_i]$ of the real axis $\mathbb{R}$. We assume that we know the number of zeros $m$ of $F(z)$ in the complex plane outside $L$. This can be achieved either through physical considerations or, better, by using the argument principle [16]. We define the function

$$M(z) = 1/F(z)$$

and we seek now the poles of $M(z)$. For the sake of simplicity, we assume at first that $F(z)$ has only one zero, $a$, outside $L$; therefore, $M(z)$ has one pole, $a$, outside $L$. We denote the corresponding residue by $C$. Hence, we have near $a$

$$M(z) = M^*(z) + C/(z - a),$$

where $M^*(z)$ is an analytic function at $a$. On the other hand, we denote the principal part of $M(z)$ at infinity by $G_\infty(z)$. This is of the form

$$G_\infty(z) = \sum_{k=0}^{\infty} A_k z^k$$

if $M(z)$ has a pole of order $l$ at infinity. Therefore,