CLASSICAL SOLUTIONS IN GRASSMANNIAN $\sigma$-MODELS

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ABSTRACT. We study the generalisation of the $\mathbb{C}P^{n-1}$ model in two-dimensional Euclidean space-time to Grassmannian $\sigma$-models having a non-Abelian gauge group. Some classes of classical solutions are displayed. It appears that the general solutions involve complicated constraints.

1. INTRODUCTION

Recently it was found [1], [2] that all finite action classical solutions of the $\mathbb{C}P^{n-1}$ model in two-dimensional Euclidean space-time could be classified in a simple way in terms of holomorphic curves. One possible application of this observation, which so far has not been fully explored, would consist of studying the effect of the collection of all stationary points on a semiclassical approximation to functional integrals, which would hopefully lead to a better approximation than the simple instanton gas approximation. Another possibility suggests investigation of whether the method of finding general classical solutions might generalise to other models of current interest. This was found to be the case in the supersymmetric extension of the $\mathbb{C}P^{n-1}$ model [3], [4], [8]. The main virtue of the general solutions of the equations of motions of the $\mathbb{C}P^{n-1}$ and $S\mathbb{C}P^{n-1}$ models is, that they are given by simple and explicit expressions in terms of the basic holomorphic curves.

A complication arises when one considers the $O(n)$ model where the solutions involve certain constraints. In this case the classification of solutions is relatively simple but less amenable to applications because of the complicated nature of the constraints.

The Grassmannian $\sigma$-model $G(n, p)$ is a simple generalization of the $\mathbb{C}P^{n-1}$ model, where the Abelian $U(1)$ gauge group is replaced by a non-Abelian $U(p)$ group. The instanton solutions of this model were studied in Reference [5], but so far, a general classification of the classical finite action solutions is still lacking (an attempt to find such solutions was made in Reference [7]). Here we will not solve the problem of finding all finite action solutions to the classical equations of motion in full generality, but we will limit ourselves to displaying explicitly some classes of such solutions and to discussing some of their properties.

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We will also point out that it is very likely that the general solutions will involve certain constraints which would render a general construction of such solutions much more implicit and, thus, less useful.

2. DEFINITION OF THE MODEL AND ITS INSTANTON SOLUTIONS

The Grassmannian $G(n, p)$ (with $p < n$) is defined in terms of a $p \times n$ matrix field $z$:

$$z_{\alpha i}(x), \quad \alpha = 1, ..., n, i = 1, ..., p$$  \hspace{1cm} (2.1)

where $x \in \mathbb{R}^2$. The field $z$ can also be thought of as being of a column of $p$ $n$-dimensional vectors $z_i, i = 1, ..., p$. The field $z$ is required to fulfill the constraint $z_i \cdot \bar{z}_j = \delta_{ij}$ or, in matrix notation,

$$zz^+ = 1_p$$  \hspace{1cm} (2.2)

and thus we see that for $p = 1$, this field reduces to the $z$ field of the $CP^{n-1}$ model. We introduce the complex variables $x_+ = x_1 \pm ix_2$ in terms of which the Lagrangian is given by

$$\mathcal{L} = 2 \text{tr} \left[ D_+ z (D_+ z)^+ + D_- z (D_- z)^+ \right]$$  \hspace{1cm} (2.3)

where

$$D_\pm = \partial_\pm - z z^+$$  \hspace{1cm} (2.4)

is the $p \times p$ matrix covariant derivative. It is evident that $\mathcal{L}$ is invariant under local $U(p)$ gauge transformations $z \to Uz, U \in U(p)$, as such a transformation leaves invariant the $p$-plane defined by the $p$ vectors $z_i, i = 1, ..., p$.

The equations of motion corresponding to (2.3) and (2.2) are

$$(D_+ D_- z + D_- z D_+ z^+) z = 0$$  \hspace{1cm} (2.5)

together with the constraint (2.2).

A particular class of solutions of (2.5) are the instanton solutions which fulfill

$$D_- z = 0$$  \hspace{1cm} (2.6)

and the anti-instanton solutions which fulfill $D_+ z = 0$. The solutions to (2.6) were found in Reference 5. Also, there the so-called instanton coordinates were introduced. They do not, however, seem to be particularly useful in finding solutions of (2.5) in the general case. Let us give a short proof of how to find the solutions of (2.6). For the matrix $z$ being a regular $p \times n$ matrix, there exists an invertible $p \times p$ submatrix $L$ of $z$ such that

$$z = L^{-1} \tilde{z}$$  \hspace{1cm} (2.7)