SOME PURELY RADIAL MOTIONS WITH DRAG
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Summary

Some academic results in the field of nonlinear mechanics are here presented. Specifically, exact solutions of the equations of motion for a body moving purely radially in nonlinear central fields of force are given, when the drag of the (homogeneous) medium is taken into account. The drag is assumed to be proportional to the velocity, and several types of forces, including gravitation, are considered. Next, for a nonhomogeneous atmosphere, we assume a density function that may represent the earth's atmosphere and a drag law consisting of two terms (linear and cubic in the velocity), and obtain the solution for a particular case.

§ 1. Introduction. We consider here some (one-dimensional) radial motions of a body in nonlinear central fields of force, firstly when the medium is a homogeneous fluid which offers a resistance to its motion proportional to the velocity, and we obtain exact solutions of the equation of motion for four types of central forces. The solutions we give do not seem to have appeared before in the literature of this field of nonlinear mechanics, and two of them exhibit the somewhat unusual feature that the argument of certain Bessel functions is real in a domain of the phase-plane, and imaginary in another domain.

With the linear drag assumed, the equation of motion is

\[ \ddot{r} + a\dot{r} + F(r) = 0, \quad a > 0 \] (1)

an essentially nonlinear differential equation, which goes over by means of the substitution \( \dot{r} = w(\dot{r}) \) into the phase-plane equation

\[ \frac{dw}{dr} + a + \frac{F(r)}{w} = 0, \] (2)

an equation whose general solution for arbitrary \( F(r) \) is unknown.

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For the case of gravitational attraction, with \( F_1(r) = b/r^2, b > 0 \), the author has been able to integrate (2) exactly, so that a quadrature gives the solution of (1) (this solution is also valid when \( b < 0 \), as in electrostatic repulsion). The first purpose of this paper is to give this solution, together with the (likewise exact) solutions for the cognate cases: \( F_2(r) = b/r \) (where the attracting or repelling body is an infinite cylinder; this may have application to ionic motion towards or away from a charged cylinder),

\[ F_3(r) = b/r, \quad \text{and} \quad F_4(r) = (b/r) + \left( b^2/a^2r^3 \right), \]

the two latter cases being only of theoretical interest.

Secondly, we give the solution for the case of a body moving radially in a nonhomogeneous medium under the influence of the gravitational attraction, when the drag is given by the law

\[ \rho(r)(ar + br^3), \quad a, b > 0 \]

where the density function, \( \rho(r) \), is of a type (decreasing with increase of \( r \)) which may approximate an actual density-variation, and where the velocity dependence involves a cubic term. This solution may be applicable, in a proper velocity range, to the case of a body moving towards the center of the earth, as well as to that of a projectile fired vertically upwards, when the rotations of the earth and the atmosphere are neglected.

\section*{§ 2. Force varying as the inverse square of the distance.}

The equation of motion is

\[ \ddot{r} + ar + \frac{b}{r^2} = 0. \]  

(3)

The \( r, \omega (=\dot{r}) \) phase-plane (for \( r > 0 \)) is divided into three open domains by means of the two curves defined by the equation

\[ r(\omega + ar)^2 = 2b. \]

The upper curve falls from \(+\infty\) at \( r = 0 \), crosses the \( r \)-axis at \( r = (2b/a^2)^{1/4} \) and becomes asymptotic to the line \(-ar\). The lower curve rises from \(-\infty\) at \( r = 0 \), reaches a (negative) maximum of ordinate \(-3(ab/2)^{1/4}\) at \( r = (b/2a^2)^{1/4} \), after which it also becomes asymptotic to the line \(-ar\). See figure 1.