SL(n + 1, C) Strata and Orbits in the Solution Space of Euclidean CPⁿ Models

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Abstract. We further study the action of SL(n + 1, C) on the space of finite action solutions of the two-dimensional Euclidean CPⁿ models. We decompose the space of k-instantons into strata. Each stratum is characterized by an integer m with 0 ≤ m ≤ min(k, n) which can be calculated from the instanton by purely algebraic means. The k-instantons with m = n are called generic. Their stratum is shown to be dense in the space of k-instantons when k ≥ n. The isotropy subgroups for each stratum are identified.

Résumé. Nous poursuivons l'étude de l'action de SL(n + 1, C) sur l'espace des solutions à action finie du modèle CPⁿ sur l'espace euclidien bi-dimensionnel. L'espace des k-instantons est décomposé en strates. Chaque strate est caractérisée par un entier m tel que 0 ≤ m ≤ min(k, n) et qui peut être calculé à partir de l'instanton par des méthodes purement algébriques. Les k-instantons avec m = n sont dits génériques. Leur strate est dense dans l'espace des k-instantons (lorsque k ≥ n). Les sous-groupes d'isotropie de chacune des strates sont identifiés.

1. In a previous paper [1], we exhibited an explicit action of the group SL(n + 1, C) on the space of finite action solutions of two-dimensional Euclidean CPⁿ models. This action reduces to the obvious global symmetry of the models when restricted to SU(n + 1), but extends to a far from trivial symmetry for the remaining part of SL(n + 1, C). It turns out that this action of SL(n + 1, C) is not transitive on the solution space; in fact, it is even not transitive on the set of k-instanton solutions for a fixed k.

In this Letter, we study the structure of strata of this action on the space of k-instantons. Using the concept of the family of a solution introduced in [1, 2], we define m(z) + 1 to be the number of solutions in the family of the solution z. (Hence 0 ≤ m(z) ≤ n.) The first proposition asserts that m(z) is invariant under the group action.

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The solutions with \( m(z) = n \) will be called \textit{generic} and all the other ones \textit{degenerate}. The second proposition proves that any degenerate solution (with \( m(z) < n \)) can be brought into a solution of the CP\(^n\) model by the action of \( \text{SL}(n + 1, \mathbb{C}) \). With the latter result, it is possible to obtain the isotropy subgroup of any solution and hence to identify the various strata. Each stratum is the set of solutions with a given \( m \). Finally we show that for \( k \geq n \), the set of generic \( k \)-instantons is dense in the set of all \( k \)-instantons, hence justifying the name \textit{generic}.

2. Let us begin by recalling some well-known facts about CP\(^n\) models defined on two-dimensional Euclidean space with coordinates \( x_{\pm} = x_1 \pm ix_2 \). The field of the model is a vector \( z(\mathbf{x}_\pm, \mathbf{x}_- \in \mathbb{C}^{n+1} \) of unit length, satisfying the field equation:

\[
D_+ D_+ z + (z^+ D_- D_+ z) z = 0,
\]

where the covariant derivatives \( D_\pm \) are defined on any field as:

\[
D_\pm = \delta_\pm - (z^\dagger \delta_\pm z).
\]

The (anti-)self-duality condition takes the simple form:

\[
D_\mp z = 0
\]

and (anti-)instantons are defined as the solutions of Equation (3) with finite action. The general \( k \)-instanton solution is given by \((n + 1)\) polynomials \( p_l(x_+) \) of \( x_+ \) only \((0 \leq l \leq n)\), with no common zero and such that the maximal degree of the \( p_l \)'s is \( k \) [2]:

\[
z_l = \frac{p_l(x_+)}{\sqrt{\sum_{l=0}^{n} |p_l(x_+)|^2}}.
\]

Define now operators \( P_\pm \) by their action on any field \( f(x_+, x_-) \in \mathbb{C}^{n+1} \) as:

\[
P_\pm f = \delta_\pm f - \left( \frac{f^\dagger \delta_\mp f}{|f|^2} \right) f.
\]

Acting on solutions \( z \) of Equation (1), the operators \( P_\pm \) have the property:

\[
|z|^2 P_- P_+ z = -z \quad |P_+ z|^2.
\]

Let now \( z \) be any finite action solution of the CP\(^n\) model.

**DEFINITION.** The family of \( z \), denoted by \( \{z\} \), is the ordered set:

\[
\{z\} = \left\{ \frac{P_-^{-1} z}{|P_-^{-1} z|}, \frac{P_+ z}{|P_+ z|}, \ldots, \frac{P_-^{-1} z}{|P_-^{-1} z|}, \frac{P_+ z}{|P_+ z|}, \ldots, \frac{P_- z}{|P_- z|}, \frac{P_+^{-1} z}{|P_+^{-1} z|}, \frac{P_+ z}{|P_+ z|} \right\}
\]

where \( i \) and \( j \) are defined as the smallest integers such that \( P_\pm^{-1} z = P_\pm^{i+1} z = 0 \). Then [2], \( i + j \leq n \). Hereafter, we will always denote the number of vectors in the family.