RELATIVISTIC STRING MODEL IN DIFFERENTIAL FORM

K. KAMIMURA

Theoretical Physics Institute, The University of Alberta, Edmonton, Alberta, Canada

ABSTRACT. The relativistic string model is reformulated in terms of differential geometry. Some independent geometrical quantities are regarded as dynamical variables instead of the coordinates. In a particular gauge the fundamental equations can be linearized and solved. The solution here found is equivalent to that of light like gauge as a classical system.

Recently the relativistic string model [1] was re-examined by using the technique of differential geometry by Omnes [2] and Barbashov et al. [3]. Omnes found that the fundamental equation which determines the structure of the world sheet of the string becomes a nonlinear one in the 3-dimensional Minkowsky space. They suggested a different mass spectrum from that of the conventional string model [1,4].

In the first part of this letter we reformulate the string model in differential form. Although the world sheet of the free string is topologically simple, the coordinate free formulation of the theory is often useful.

In the second part we show the fundamental equation can be linearized in a particular gauge in contrast to the nonlinear one obtained in Refs. 2 and 3. The coordinate $X^\mu$ is explicitly found by integrating the differential forms. This classical solution is essentially the same as that used in the definite metric quantization of the string model [4].

The world sheet $\Sigma$ of the string is a two-dimensional time-like minimal surface in the Minkowsky space. $\Sigma$ is bounded in the space-like direction and the boundary condition is that there is no momentum flow from the boundary $\partial \Sigma$ of $\Sigma$.

This dynamical system is described by the action

$$I = - K \text{ (area of } \Sigma) = \frac{K}{2} \int_\Sigma dX^\mu \ast dX_\mu.$$  \hspace{1cm} (1)

The invariance of $I$ under the variation of $X^\mu$ gives the equation of motion

$$Kd \ast dX = 0 \text{ \hspace{1cm} (on } \Sigma)$$  \hspace{1cm} (2)

under the boundary condition

$$K \ast dX^\mu = 0 \text{ \hspace{1cm} (on } \partial \Sigma).$$  \hspace{1cm} (3)
\( K \ast dX^\mu \equiv dP^\mu \) is a momentum one-form and the total momentum and the angular momentum are defined by

\[
\mathcal{P}_\mu \equiv - \int_\Phi dP_\mu
\]

\[
\mathcal{M}_{\mu\nu} \equiv - \int_\Phi X_{[\mu} dP_{\nu]}\]

where \( \Phi \) is the space-like cross-section of \( \Sigma \). These are shown to be independent of \( \Phi \) by (2) and (3).

In the conventional approach \( X^\mu \) is regarded as dynamical variables, then the Lagrangian is singular as a result of the local coordinate (parameter) transformation of \( \Sigma \). In the geometrical approach some geometrical quantities defined in the following [5] are regarded as more fundamental.

First of all, \( dX^\mu \) is a tangential vector of \( \Sigma \) with one-form coefficients. In terms of the frame

\[
dX^\mu = \sum_{a=0}^1 \sigma^a e_a^\mu
\]

where \( e_\alpha \) is a time-like and \( e_I \) is a space-like unit tangential vector of \( \Sigma \). By introducing two normal vectors \( e_f^\mu (f = 2, 3) \), \( e_\alpha^\mu \)s (\( \alpha = 0, 1, 2, 3 \)) form a local triad

\[
e_\alpha^\mu e_{\beta\mu} = \eta_{\alpha\beta} = (+; -; -; -).
\]

Hereafter the indices \( a, b, ... \) take the values 0 and 1; \( i, j, ... \) take the values 2 and 3; and \( \alpha, \beta, ... \) take the values 0, 1, 2, and 3. The summation of repeated indices is understood. \( d e_\alpha^\mu \) is also expanded by \( e_\rho^\mu \) with one-form coefficients

\[
d e_\alpha^\mu = \Omega_\alpha^\beta e_\rho^\mu.
\]

Due to (6) \( \Omega_\alpha^\beta \equiv \Omega_\alpha^\gamma \eta_{\gamma\beta} \) is anti-symmetric

\[
\Omega_\alpha^\beta + \Omega_\beta^\alpha = 0.
\]

Equations (5), (7) and (8) are the structure equations. In order that these forms are integrable the following condition must be satisfied (Poincaré Lemma)

\[
d \sigma^a = \sigma^b \Omega_b^a
\]

\[
\sigma^a \Omega_a^I = 0
\]

\[
d \Omega_\alpha^\beta = \Omega_\alpha^\gamma \Omega_\gamma^\beta.
\]