DISSIPATIVE OPERATORS AND COHOMOLOGY OF OPERATOR ALGEBRAS

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ABSTRACT. It is proved that an ultraweakly continuous completely dissipative operator on a W*-algebra Ŕ has a canonical form in terms of a completely positive map and a Hamiltonian provided that the cohomology groups of Ŕ with coefficients in a dual normal Ŕ-module are zero.

1. INTRODUCTION

The object of this note is to indicate a connection between the existence of a canonical form for the completely dissipative operators on a W*-algebra Ŕ, as proved in [1] for a special case, and the vanishing of the cohomology groups of Ŕ with coefficients in certain Ŕ-modules.

The formal similarity of this work with that of Parthasarathy and Schmidt on conditionally positive definite functions on certain topological groups should be evident [2].

We first recall some notations and facts from the cohomology theory of C* and W*-algebras [3, 4, 5, 6].

If Ŕ is a C*-algebra and Ŕ a twosided Banach Ŕ-module, then the set of n-cochains C^n(Œ, Ŕ) on Ŕ with values in Ŕ is the space of separately continuous n-linear maps x d-*.lt' and C^0 (Œ, Ŕ) = Ŕ (sometimes the notation C^o (Œ, Ŕ) is used). The coboundary map A : C^n (Œ, Ŕ) \rightarrow C^{n+1} (Œ, Ŕ) is defined by

$$A F (X_1, \ldots, X_n) = X_1 F (X_2, \ldots, X_n) + \sum_{i=1}^{n-1} (-1)^i F (X_1, \ldots, X_i X_{i+1}, \ldots, X_n) + (-1)^n F (X_1, \ldots, X_{n-1}) X_n.$$ 

A satisfies the relation A^2 = 0. Define

$$Z^n (Œ, Ŕ) = \{ F \in C^n (Œ, Ŕ) ; A F = 0 \}$$

$$B^n (Œ, Ŕ) = \{ F = \Delta G ; G \in C^{n-1} (Œ, Ŕ) \} \subset Z^n (Œ, Ŕ)$$

$$H^n (Œ, Ŕ) = Z^n (Œ, Ŕ) / B^n (Œ, Ŕ).$$

H^n (Œ, Ŕ) is called the n-dimensional cohomology group of Ŕ with coefficients in Ŕ. 

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\( \mathcal{M} \) is called a dual \( \mathcal{A} \)-module if it is the dual space of a Banach space \( \mathcal{M} \) and if for each \( X \in \mathcal{A} \) the maps \( \mathcal{M} \rightarrow \mathcal{M} : \xi \rightarrow X\xi, \xi \rightarrow \xi X \) are weak * continuous [3].

\( \mathcal{M} \) is called a dual normal \( \mathcal{A} \)-module if in addition for each \( \xi \in \mathcal{M}, X \rightarrow \xi \) and \( X \rightarrow \xi X \) define ultraweak- weak * continuous maps \( \mathcal{A} \rightarrow \mathcal{M} \). The set of \( F \rightarrow C^n(\mathcal{A}, \mathcal{M}) \) which are separately ultraweak- weak * continuous are denoted by \( C^n_{uw}(\mathcal{A}, \mathcal{M}) \) in [3].

We have the following useful results.

\[
H^{n+P}(\mathcal{A}, \mathcal{M}) \cong H^n(\mathcal{A}, C^p(\mathcal{A}, \mathcal{M}))
\]

with a suitable \( \mathcal{A} \)-module structure on \( C^p(\mathcal{A}, \mathcal{M}) \) [3].

If \( \mathcal{A} \) is a W*-algebra and \( \mathcal{M} \) a dual normal \( \mathcal{A} \)-module then [5]

\[
H^n_{uw}(\mathcal{A}, \mathcal{M}) \cong H^n(\mathcal{A}, \mathcal{M}).
\]

If \( \mathcal{A} \) is a W*-algebra of type I or hyperfinite and \( \mathcal{M} \) a dual normal \( \mathcal{A} \)-module then

\[
H^n(\mathcal{A}, \mathcal{M}) = 0
\]

for \( n = 1 \), hence for all \( n \) [5].

2. DISSIPATION FUNCTIONS

Let \( \mathcal{A} \) be a C*-algebra with unit and consider a norm continuous semigroup of completely positive maps \( \mathcal{A} \rightarrow \mathcal{A} \) as described in [1]:

\[
\Phi_t \in CP(\mathcal{A})
\]

\[
\Phi_s \cdot \Phi_t = \Phi_{s+t}
\]

\[
\Phi_t(I) = I
\]

\[
\lim_{t \to 0} \| \Phi_t - I \| = 0.
\]

The complete positivity of \( \Phi \) means that

\[
\sum_{i,j} Y_i^* \Phi(X_i^*X_j) Y_j \geq 0
\]

for all \( \{X_i, Y_j\} \subseteq \mathcal{A} \) and all \( n \). From this follows the Schwarz type inequality

\[
\sum_{i,j} Y_i^* \left[ \Phi(X_i^*X_j) - \Phi(X_i^*) \Phi(X_j) \right] Y_j \geq 0
\]

(2.1)

by doubling the index set and making the replacements \( \{X_i\} \rightarrow \{X_i, -I_i\}, \{Y_i\} \rightarrow \{Y_i, -\Phi(X_i)\} \).