THE SYMPLECTON: A PROTOTYPE FOR SEMI-SIMPLE GRADED LIE ALGEBRAS†

L. C. BIEDENHARN* and J. D. LOUCK

Theoretical Division, Los Alamos Scientific Laboratory,
University of California, Los Alamos, New Mexico 87545

ABSTRACT. An elementary, physically motivated, example of a semi-simple graded Lie algebra (SSGLA) is shown to be given by the 'symplecton realization' of angular momentum [an associative, involutive, inner-product algebra whose characteristic polynomials realize the symplectic group Sp(2)]. The Pais-Rittenberg result shows that the $n$-component symplecton realizes the most general, SSGLA, Sp(2n).

Invariance techniques in quantum physics have been significantly extended in recent years by the concept of 'supersymmetry' (and 'supergauges') introduced by Wess and Zumino [1], and by Volkov and Soroka [2]; supersymmetry involves linear transformations relating states of different statistics, with supermultiplets combining, in general, both bosons and fermions. The concept of supersymmetry was recently clarified by Corwin, Ne'eman and Sternberg [3] who demonstrated that supersymmetry was properly subsumed under the mathematical construct of graded Lie algebras (GLA), a topic well studied in the mathematical literature. Still more recently, Pais and Rittenberg [4] have considered semi-simple GLA, and arrived at a complete classification.

The purpose of the present note is to show that the 'symplecton' — a model introduced prior to supersymmetry as realizing primitively angular momentum SU(2) symmetry — is, from the work of Pais-Rittenberg, precisely the prototype for all semi-simple GLA. We will show below that the symplecton construction not only simplifies greatly their work, but has the considerable advantage of providing an easily understood model, taken from the quantum theory of angular momentum (QTAM), which, we believe, will make the relevant concepts accessible to a wider audience.

Let us recall the formal definition of a GLA. One has a graded vector space: \( L \equiv \bigoplus_{k} L_{k} \), that is, \( L \) is a vector space, whose elements are finite direct sums of components lying in the vector spaces \( L_{k} \), and the index \( k \) (taken to be an integer here) belongs to a finite Abelian group. Then \( L \) is a graded Lie algebra if we have a bilinear map [5]: \( [L, L] \rightarrow L \) such that:

\[
[L_{k}, L_{\bar{k}}] \subset L_{k+\bar{k}}; \tag{1}
\]

\[
[x, y] = (-)^{k\bar{k}+1} [y, x]; \tag{2}
\]

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\[ [x, [x, z]] = [[x, y], z] + (-)^k \delta \delta [y, [x, z]]; \tag{3} \]

where \( x \in L_k, y \in L_k \).

It is clear from (2) that both commutators and anti-commutators will enter the general case.

The symplecton construction arose in answer to the question: What is the simplest conceivable way to realize QTAM? The Jordan-Schwinger realization in terms of a two-state boson \((a_1, a_2)\) is certainly simple, but, in fact, one may realize QTAM on a single boson \(a\) (with \([a, a] = 1\)) — this is the symplecton realization [6]. To accomplish this, one must not use the usual boson rule for the vacuum (i.e., \(a|0\rangle \neq 0\) now); states are accordingly polynomials over \(a\) and \(\bar{a}\), with complex coefficients, implemented with the commutation rule: \([\bar{a}, a] = 1\). The use of bosons to construct half-integer spin states clearly indicates a formal supersymmetry.

The essential content of this construction can be easily understood from the basic result [7]. There exists a family of characteristic polynomials \(P^m_j\) which obey the product law:

\[ P^\alpha \beta \gamma_{\alpha \beta \gamma} = \sum_{c, \gamma} C^{bac}_{\beta \alpha \gamma} P^\gamma, \tag{4} \]

where

\[ \langle c | a | b \rangle = (2c + 1)^{-\frac{1}{2}} \cdot \Delta(abc), \tag{4a} \]

\[ \Delta(abc) = \left[ \frac{(a+b+c+1)!}{(A+b-c)!(a-b+c)!(A-a+b+c)!} \right]^{\frac{1}{2}}, \tag{4b} \]

and \(C^{\beta \alpha \gamma}_{\beta \alpha \gamma}\) is the usual Wigner ('Clebsch-Gordan') coefficient for SU(2).

The most interesting aspect of this construction (aside from the fact that it can be done at all) is that the triangle function \(\Delta\) can be seen to belong the special functions of QTAM, and could indeed be called the \((3j)\) function. It is a non-trivial consequence of the associativity of multiplication in this algebraic structure which implies that the \(\Delta\) transform by a Racah relation [6], analogously to the Wigner and Racah functions themselves.

A second important property of the \(P^m_j\) is the existence of an adjoint: \(P^m_j_{\alpha \beta} = (-)^{\alpha - a} P^m_{\alpha \beta}\). Using this, and taking the \(P^m_{\alpha \beta}\) as a basis, one defines an inner product:

\[ \langle aa | b \beta \rangle \overset{\text{def}}{=} \text{the } c = 0 \text{ component of } (P^m_{\alpha \beta})_{\alpha \beta} \]

\[ = \delta_{\alpha}^a \delta_{\beta}^b. \]

That this structure has the desired properties of an inner product depends critically on the properties of the Wigner coefficients. (Alternatively [8] one could define the Wigner coefficients intrinsically by the trilinear invariant \(\Delta\).)

Denoting a general element by \(x = \sum_{\alpha \beta} x_{\alpha \beta} \in \mathbb{C}\), we see that the \(\{x\}\) have the algebraic structure of a ring equipped with a norm [9] and an involution \((*: x \rightarrow x^*, \text{with } x_{\alpha \beta} \rightarrow x_{\alpha \beta}, P_{\alpha} \rightarrow P_{\alpha}^*\).\)

It is not difficult to check now the defining properties (1), (2), (3) above, and verify that the symplecton structure is indeed a GLA. To do so, one first splits the eigenpolynomials into the two