Short- and Long-Time Behavior of Eddy-Viscosity Models

Leslie M. Smith and Victor Yakhot
Applied and Computational Mathematics Program,
Princeton University, Princeton, NJ 08544, U.S.A.

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Abstract. The short-time behavior of the turbulent viscosity is inferred from the immediate response of the Reynolds stress deduced by Crow [1] for the problem of isotropic turbulence subjected to a mean strain at time \( t = 0 \). The turbulent viscosity \( v \) is defined for \( t \rightarrow 0 \) by the relation \( T_{ij} = -2vS_{ij} \), where \( T_{ij} \) is the Reynolds stress and \( S_{ij} \) is the mean rate of strain. It follows that the viscosity is \( v = O(t) \) for \( t \rightarrow 0 \). Matching the short- and long-time behaviors, we propose an analytic expression for the effective viscosity valid for all time. Using the proposed viscosity, the \( \mathcal{K} - \mathcal{E} \) model for homogeneously sheared turbulence is reformulated to be valid in both the short- and long-time limits. Previously, the \( \mathcal{K} - \mathcal{E} \) model has been used with the long-time form of the effective viscosity for all time. Comparison of theoretical predictions with the results of physical and numerical experiments is presented. Implications of the short-time response for large-eddy simulations and spectral-space closure theories are discussed.

1. Introduction

The concept of the turbulent viscosity plays a central role in the theory of turbulence and turbulence modeling. The origin of the concept can be traced to Richardson [2] and Prandtl [3], who perceived turbulence as a gas in which eddies play the role of molecules. In kinetic theory the molecular viscosity is given by \( \nu_0 = v_{rms}^2 \), where \( v_{rms} \) is the root-mean-square velocity of thermal fluctuations and \( \lambda \) is the mean free path. This expression is valid only for correlation times \( t > \tau_0 \), where \( \tau_0 = O(2/v_{rms}) \).

At shorter times \( t \ll \tau_0 \), the viscosity is \( \nu_0 = O(v_{rms}^2 t) \), which can be derived from the Kubo formula [4].

By analogy, in turbulent flow one can define \( \nu \approx v_{rms}^2 \tau_L \), where \( v_{rms} \) is the mean value of the turbulent velocity fluctuations. The time \( \tau_L \) is proportional to the eddy turnover time, \( \tau_L = O(\mathcal{K}/\mathcal{E}) \), where \( \mathcal{K} \) and \( \mathcal{E} \) are the local values of the mean kinetic energy and mean dissipation rate of energy, respectively. Using \( v_{rms}^2 = \mathcal{K} \) and \( \tau_L = O(\mathcal{K}/\mathcal{E}) \) leads to the formula

\[
\nu = A \frac{\mathcal{K}^2}{\mathcal{E}},
\]

where \( A \) is constant, which has been widely used in turbulence modeling. Again this expression is valid only for \( t > \tau_L \). However, for many problems of hydrodynamics we need to understand the immediate response of turbulence to instantaneously applied perturbations such as external forces and mean shear.

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Since (1) is incorrect for short-time response, it has been traditionally believed that the $\mathcal{K} - \mathcal{D}$ model cannot be used to simulate transient flows, and that a full Reynolds stress model must be resorted to. However, as we will show, the $\mathcal{K} - \mathcal{D}$ equations can indeed model nonequilibrium turbulence when the eddy viscosity (1) is modified to account for the short-time response correctly.

The short-time turbulent viscosity $\nu = O(t)$ can be inferred from the work of Crow [1]. Crow considered the response of isotropic turbulence to a mean strain applied at time $t = 0$. He showed that the Reynolds stress $T_{ij}(t) = \langle \overline{v_i v_j} \rangle - 2\mathcal{K} \delta_{ij}/3$ is given by

$$T_{ij}(t) \sim -\frac{8}{15} \mathcal{K} S_{ij} t, \quad t \to 0,$$

where $S_{ij} = \frac{1}{2} (V_j U_i + V_i U_j)$ is the mean rate of strain tensor, $V_i \equiv \partial / \partial x_i$, and the overbar denotes the mean value. Defining $\nu(t \to 0)$ by $T_{ij}(t) = -2\nu S_{ij}$ leads to

$$\nu \sim \frac{8}{15} \mathcal{K} t, \quad t \to 0.$$

The short-time behavior $\nu = O(t)$ can also be derived using perturbation analysis of the Navier-Stokes equations. Elsewhere [5] we show that regularization of Kraichnan’s Direct Interaction Approximation [6], [7] (DIA) leads to a theory for $\nu$ valid for all time, with short-time behavior consistent with (3) and long-time behavior (1). Here, however, we simply match (1) with (3), and suggest an analytic formula for $\nu$ which can be used for $\mathcal{K} - \mathcal{D}$ transport simulations of response to external shear. Our formula for $\nu$ illustrates the difference between short- and long-time behavior, but is applicable only to the problem of response to constant strain and is thus preliminary in nature.

It is easy to construct an ansatz for $\nu$ with short-time behavior (3) and long-time behavior (1). One such model is

$$\nu(\mathcal{K}, \mathcal{D}, t) = A \frac{t^2}{\mathcal{D}} \left[ 1 - \exp \left( -B \frac{t}{\mathcal{K}} \right) \right],$$

where (3) requires $AB = \frac{8}{15}$. Since we build upon the $\mathcal{K} - \mathcal{D}$ model derived using RG theory and the $\varepsilon$-expansion [8]-[10], we use the RG value of the long-time coefficient $A = 0.0854$. The limiting behaviors of (4) for $t \to 0$ and $t \to \infty$ have both been derived directly from the Navier-Stokes equations [1], [8].

In Section 2 we show the effect of the short-time response in the $\mathcal{K} - \mathcal{D}$ equations for homogeneous shear flow, and compare our results to numerical [11] and experimental data [12]-[14]. It is argued in Section 3 that the Smagorinsky [15] subgrid model is inadequate for large-eddy simulation of short-time behavior. In Section 4 the implication of the short-time response for closure theories is discussed. Section 5 compares the Lagrangian autocorrelation function following from (4) to numerical [16] and experimental [17] data. Conclusions are given in Section 6.

2. The $\mathcal{K} - \mathcal{D}$ Equations for Homogeneous Shear Flow

2.1. The $\mathcal{K} - \mathcal{D}$ Model Equations

RG and the $\varepsilon$-expansion have been used [8]-[10] to derive long-time model transport equations for the mean kinetic energy $\mathcal{K}$ and mean dissipation rate of energy $\mathcal{D}$ in turbulent flow. In the high Reynolds number limit away from walls,

$$\frac{d\mathcal{K}}{dt} + U_i \nabla_i \mathcal{K} = \mathcal{P}_x - \mathcal{D} + \nabla_i (\nu \nabla_i \mathcal{K}),$$

$$\frac{d\mathcal{D}}{dt} + U_i \nabla_i \mathcal{D} = C_{\varepsilon_1} \mathcal{P}_x \frac{\varepsilon}{\mathcal{K}} - C_{\varepsilon_2} \frac{\mathcal{D}^2}{\mathcal{K}} - \mathcal{R} + \nabla_i (\nu \nabla_i \mathcal{D}),$$

$$\mathcal{P}_x \equiv -T_{ij} S_{ij}, \quad T_{ij} \equiv \overline{v_i v_j} - \frac{2}{3} \mathcal{K} \delta_{ij} = -2\nu S_{ij}, \quad S_{ij} \equiv \frac{1}{2} (V_j U_i + V_i U_j),$$

$$\mathcal{R} = 2\nu \frac{S_{ij} \partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j},$$

where $U$ is the mean velocity, $\nu$ is the fluctuating velocity, $\nu_o$ is the molecular viscosity, $V_i \equiv \partial / \partial x_i$ and...