ON A POSSIBLE EMPIRICAL MEANING OF MEETS AND JOINS FOR QUANTUM PROPOSITIONS

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ABSTRACT. Following up an idea of Jauch, an empirical basis is worked out for the meets (and joins) of quantum propositions through approximate measurement to an arbitrary degree of accuracy of special yes-no effects constructed from chains of yes-no measurements. This is done with a view to interpreting the meets (and joins) as Dedekind's cuts in analogy with irrationals.

As is well known, the quantum logic approach to axiomatic quantum mechanics is an important and much elaborated [1-4] way of seeking the answer to the inescapable questions: Where does the Hilbert space come from? What empirical evidence makes it necessary? The approach is based on the fact that $\mathcal{P}(\mathcal{H})$, the set of all projection operators in a separable Hilbert space $\mathcal{H}$, turns out to be a very special poset (partially-ordered set) with respect to the ordering relation $E \leq F \iff EF = E, E, F \in \mathcal{P}(\mathcal{H})$. This poset is a complete, separable, orthomodular atomic lattice with a covering property.

The complete-lattice property of $\mathcal{P}(\mathcal{H})$ asserts that every subset $\{E_m : m \in M\} \subseteq \mathcal{P}(\mathcal{H})$ (M being an index set) has a join (alias, the least upper bound or supremum) $\bigvee_{m \in M} E_m$ and a meet (alias, the greatest lower bound or infimum) $\bigwedge_{m \in M} E_m$. It was suggested (see sections 5-3 in [4]) that for finite $M$ the meet could be empirically based on a suitable infinite sequence of filters for the propositions $E_m$. "Such infinite processes, though not possible in actual physical measurement, can be used as a base for an approximate determination of $\bigwedge_{m \in M} E_m$ to any needed degree of accuracy."

In a Jauch–Piron conceptual framework that uses only propositions (and underlying yes-no measurements) as primitive axiomatic entities, there are apparently insufficient basic concepts to give the mentioned claim a precise and quantitative meaning 'to any needed degree of accuracy'.

It is the aim of this letter to show that:
(i) Besides utilizing $\mathcal{P}(\mathcal{H})$, also the statistical operators (states) in $\mathcal{H}$, Jauch's arbitrary-degree-of-accuracy idea can be easily given a precise meaning (in principle at least);
(ii) this can be extended to $M$ becoming countably infinite (in contrast to Jauch's claim);
(iii) the case of $M$ being noncountably infinite can be reduced to (i) or (ii) owing to separability;
(iv) the meaning of (i)-(iii) can be interpreted via Dedekind's cuts in analogy with exactly-determinable rationals and only approximately determinable irrationals on the real axis.
(i) **Finite meets.** Let \( E_1, E_2, \ldots, E_n \in \mathcal{P}(\mathcal{H}) \), and let \( E_n \circ E_{n-1} \circ \cdots \circ E_2 \circ E_1 \) ("\( \circ \)" to be read as 'after') be a chain of filters corresponding to the projection operators, beginning with \( E_1 \) and ending with \( E_n \). After the last filter, one has a detector recording the yes results in an ensemble of measurements. It is a yes-no chain arrangement, but not a yes-no measurement (unless all \( n \) operators commute) [5, 6]. It has no repeatability: having got a yes result (when in the last step we have a first-kind or predictive yes-no measurement for \( E_n \)), an immediate repetition with an analogous chain arrangement does not necessarily give yes again (except in the commuting case). Theoretically, in general, there is no element in \( \mathcal{P}(\mathcal{H}) \) corresponding to this experiment. We shall call it a yes-no effect.

**Lemma 1.** Let \( \rho \) be an arbitrary statistical operator in \( \mathcal{H} \). The probability of a yes result in measuring the yes-no effect \( E_n \circ E_{n-1} \circ \cdots \circ E_2 \circ E_1 \) in \( \rho \) is

\[
\text{Tr}(E_1 E_2 \cdots E_{n-1} E_n E_{n-1} \cdots E_2 E_1) \rho.
\]  

Proof. To prove Equation (1) by total induction, we make the additional claim that the ensemble of quantum systems obtained by selecting out those with a yes result in a first-kind version (see above) of the measurement at issue (the yes ensemble) is described by

\[
\rho_n = \frac{E_n E_{n-1} \cdots E_2 E_1 \rho E_1 E_2 \cdots E_{n-1} E_n}{\text{Tr}(E_1 E_2 \cdots E_{n-1} E_n E_{n-1} \cdots E_2 E_1)} \rho.
\]

Expression (1) and equality (2) are well known as being valid for \( n = 1 \) and we assume, for the moment, their validity for \( n \). Then one can view the probability of \( E_{n+1} \circ E_n \circ \cdots \circ E_1 \) as the product of the conditional probability \( \text{Tr} E_{n+1} \rho_n \) and the probability given by (1) (that of the condition). This gives (1) with \( n \) replaced by \( n + 1 \). The yes-ensemble corresponding to \( E_{n+1} \circ E_n \circ \cdots \circ E_1 \) is described by \( E_{n+1} \rho_n E_{n+1} / \text{Tr} E_{n+1} \rho_n \). And this is \( \rho_{n+1} \), i.e., it is of the form (2) with \( n \) replaced by \( n + 1 \). Hence, (1) and (2) are valid for all integers \( n \).

**Lemma 2.** Let

\[
F_n = E_1 E_2 \cdots E_{n-1} E_n E_{n-1} \cdots E_2 E_1.
\]

Then

\[
\lim_{k \to \infty} F_n^k = E_1 \wedge E_2 \wedge \cdots \wedge E_{n-1} \wedge E_n
\]  

(convergence in the strong sense).

Proof. Since \( 0 \leq F_n \leq 1 \), multiplication with \( F_n^k \) gives \( 0 \leq F_n^{k+1} \leq F_n^k \), \( k = 1, 2, \ldots \) [7]. Hence, \( \{F_n^k : k = 1, 2, \ldots \} \) is a monotonously decreasing bounded sequence and, thus, it converges in the strong sense to a positive operator [7].

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