A CLASSICAL SOLUTION FOR MANY-BODY FORCES
OF DIRECTLY INTERACTING PARTICLES

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Abstract. The first class constraint conditions of relativistic direct interaction dynamics have been solved. Their solution expresses the many-body interactions in terms of the two-body interactions in perturbation expansion. The n-body interaction is of order g smaller than the (n - 1)-body interaction. The expressions are considerably simpler than those obtained previously.

1. Introduction

In recent years, the relativistic dynamics of closed systems of N mutually directly interacting particles has been developed. On the classical level, difficulties arise when Poincaré invariance is required from a dynamics based on a symplectic space [1]. These difficulties are removed by enlarging the dimensionality of the symplectic space and imposing suitable constraints to ensure the correct number of degrees of freedom. The suggestion of using generalized mass shells as first class constraints [2] is especially attractive. For N spinless particles labelled a (a = 1, 2, ..., N) these are

\[ K_a = p_a^2 + m_a^2 + V_a + W_a \approx 0. \]

Here \( p_a \) is the momentum of the \( a \)th particle, \( m_a \) its mass, \( V_a \) the sum of all two-body interactions acting on particle \( a \), and \( W_a \) the sum of all \( n \)-body interactions of particle \( a \) (\( n = 3, 4, ..., N \)). These \( N \) constraints must eliminate \( 2N \) degrees of freedom. This can be ensured by requiring them to be first class, i.e.,

\[ \{ K_a, K_b \} \approx 0 \quad \forall a, b. \]

For a system consisting only of particles \( a \) and \( b \), (2) requires

\[ \{ p_a^2, V_b \} = \{ p_b^2, V_a \}. \]

This simplifies the requirement (2) for the case \( N \geq 2 \) and yields the set of simultaneous equations
which we call the first-class constraints condition. Here we use the more restricting strong equality to obtain explicit solutions. It has been shown that this condition has only trivial solutions unless the $w_a$ do not vanish, i.e., unless many-body interactions are present [3]. These equations therefore are conditions which the many-body interactions must satisfy when the two-body interactions are given. Various solutions have been proposed recently [4, 5]. However, the need for many-body interactions in systems where the interaction is not mediated by a field has been recognized long ago [6, 7, 8].

2. PERTURBATION EXPANSION

The first class constraint condition (4) is a set of non-linear coupled partial differential equations. Since a closed form solution has so far escaped us, we shall make the same plausible assumption as in references [4] and [5], i.e., that the $v_a$ are of order $g$, the two-body coupling constant, and the $w_a$ are of order $g^2$. Such an assumption is also suggested by quantum field theories. More precisely, we expand the $w_a$,

$$w_a = \sum_{n=2}^{\infty} w_a^{(n)}$$

where $w_a^{(n)}$ is of order $g^n$. This reduces Equation (4) as in Reference [5] to

$$\{p_{a}^{2}, W_b^{(n)}\} - \{p_{a}^{2}, W_{a}^{(n)}\} + 2C_{ab}^{(n-1)} = 0, \quad \forall a, b, \quad n = 2, 3, \ldots$$

where

$$C_{ab}^{(1)} = \frac{1}{2} \{V_a, V_b\}, \quad C_{ab}^{(2)} = \frac{1}{2} (\theta_{ab}^{(2)} - \theta_{ba}^{(2)}),$$

$$C_{ab}^{(n)} = \frac{1}{2} (\theta_{ab}^{(n)} - \theta_{ba}^{(n)} + X_{ab}^{(n)}),$$

$$E_{ab}^{(n)} = \{V_a, W_b^{(n)}\}, \quad X_{ab}^{(n-1)} = \sum_{k=2}^{n-2} \{w_a^{(k)}, w_a^{(n-k)}\}.$$  

The non-linear set (4) thus becomes a set of linear inhomogeneous coupled partial differential equations. The inhomogeneous parts of the equations for the $W^{(n)}$ are the $C^{(n-1)}$ which depend only on $W^{(k)} (k < n)$; hence all $W^{(n)}$ can be found by an iterative algorithm from the $C^{(1)}$. These depend only on the two-body interactions so that the algorithm leads to the $W$ in terms of the $V$. We can write Equation (6) as

$$p_a \cdot \frac{\partial}{\partial q_a} W_b^{(n)} - p_b \cdot \frac{\partial}{\partial q_b} W_a^{(n)} = C_{ab}^{(n-1)}.$$  

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