GOLDSTONE THEOREM FOR BOSE SYSTEMS

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ABSTRACT. In three or more dimensions ($v \geq 3$) it is proved that if the correlations decay faster than $|x|^{-\alpha}$, $\alpha = v - 2$, then gauge symmetry breaking is excluded. In one and two dimensions ($v = 1$ or 2) the gauge symmetry is always preserved.

Recently there has been a renewed interest in the Goldstone theorem within mathematical physics (see [1] and [2], and references therein). The Goldstone theorem states that an equilibrium state in which there is spontaneous breaking of a continuous symmetry must be slowly clustering. The cited works mainly treat quantum lattice systems or classical systems.

Reference [1] limits itself to a formal discussion of the Goldstone theorem for continuous Bose systems without giving definite results. In this note we want to concentrate on the Bose system with gauge symmetry. We prove that in three and more dimensions ($v \geq 3$) the breakdown of gauge symmetry implies a clustering not faster than $|x|^{-\alpha}$, $\alpha = v - 2$. This result is optimal since in the free Bose gas this is exactly the behaviour of the clustering below the condensation temperature [3]. It is interesting to note that the same behaviour is found to hold for lattice systems [4]. In one and two dimensions we prove the full gauge invariance of the state, generalizing the well-known result of Hohenberg [5] where it is only proved that the one-point function or the order parameter vanishes. To derive our results we make use of the Bogoliubov inequality as in [1] and the idea of an approximate invariance of the Hamiltonian under local gauge transformations [6, 7].

We consider the usual framework of Bose systems on $\mathbb{R}^v$. The algebra of observables is $\mathcal{A} = \mathcal{A}_\Lambda$, where $\Lambda$ is any open, connected, bounded region of $\mathbb{R}^v$, and $\mathcal{A}_\Lambda = \mathfrak{B}(L^2(\Lambda))$, $L^2(\Lambda)$ is the Fock space of symmetric functions of $L^2(\Lambda)$ with support in $\Lambda$.

On each local Fock space we take the Hamiltonian $H_\Lambda$ which on the $n$-particle subspace is given by

$$H_\Lambda^n = T_\Lambda^n + U(x_1, \ldots, x_n)$$

where

$$T_\Lambda^n = \sum_{j=1}^{n-1} - \Delta_j$$

and $U$ is such that $H_\Lambda$ is self-adjoint with Dirichlet boundary conditions.

Let $\omega$ be a translation invariant state of $\mathcal{A}$. From now on suppose that it satisfies the following three conditions

(i) The Bogoliubov inequality in the following sense

$$|\omega([x,y])|^2 \leq \frac{\beta}{2} \omega(xx^* + x^*x) \lim_{\Lambda} \omega([y^*, [H_\Lambda, y]])$$

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for all $x, y \in \mathcal{A}_\Lambda$ with $\Lambda_0 \subset \Lambda$ and $y \in \mathcal{D}([H, \cdot])$, $\beta = 1/kT$.

(ii) Local normality: For all $\Lambda$ there exists a density matrix $\rho_\Lambda$ on $\mathcal{F}(L^2(\Lambda))$ such that

$$\omega(A) = \text{tr} \rho_\Lambda A; \quad A \in \mathcal{A}_\Lambda.$$ 

(iii) The property of finite density: let $N_\Lambda = d\Gamma_\Lambda(1)$ where for any $A \in \mathcal{F}(L^2(\Lambda))$, $d\Gamma_\Lambda(A)$ on the $n$-particle subspace is defined to be

$$A \otimes 1 \otimes 1 \ldots \otimes 1 \otimes A \otimes 1 \otimes 1 \ldots \otimes 1 + \ldots$$

We suppose that for each $\Lambda$ there exists a complete orthonormal set of eigenvectors $(\phi_i)_i$ of $\rho_\Lambda$ with eigenvalue $\rho_\Lambda$ such that $\phi_i \in \mathcal{D}(N_\Lambda)$ and such that

$$\sum_i \rho_\Lambda^i(\phi_i, N_\Lambda \phi_i) < \infty$$

where $\rho$ is a given density and $|\Lambda|$ is the volume of $\Lambda$.

Note that for all $A = A^* \in \mathcal{F}(L^2(\Lambda))$, $d\Gamma_\Lambda(A) \leq \|A\|N_\Lambda$. Therefore it follows that each $\phi_i \in \mathcal{D}(d\Gamma_\Lambda(A))$ and that

$$\sum_i \rho_\Lambda^i(\phi_i, d\Gamma_\Lambda(A)\phi_i) < \infty.$$ 

Hence

$$\omega(d\Gamma_\Lambda(A)) = \lim_{t \to \infty} \left( \frac{e^{itd\Gamma_\Lambda(A)} - 1}{it} \right) = \sum_i \rho_\Lambda^i(\phi_i, d\Gamma_\Lambda(A)\phi_i).$$

In particular we consider the multiplication operators by functions $f \in C_c(\mathbb{R}^d)$ (continuous functions of compact support) which we continue to denote by $f$.

The linear functional $f \in C_c(\mathbb{R}^d) \to \omega(d\Gamma(f))$ where $d\Gamma(f) = d\Gamma_\Lambda(f)$ with $\Lambda \supset f$, is positive and therefore, in the Riesz representation theorem [6], it is represented by a unique positive measure. Since the functional is also translation invariant the measure is equivalent to the Lebesque measure and by (2):

$$\omega(d\Gamma(f)) = \rho \int dx f(x)$$

**Theorem 1.** Let $\nu \geq 3$ and suppose that the state $\omega$ satisfies the conditions (i)-(iii). Assume there is a weakly dense gauge invariant *-subalgebra $\mathcal{A}_\omega$ of $\mathcal{A}$ such that every $A \in \mathcal{A}_\omega$ satisfies the cluster property: $|\omega(A_x A) - \omega(A)|^2 \leq C(1 + |x|)^{\alpha} C > 0, \alpha > 0$ where $A_x$ is the translation of $A$ by $x \in \mathbb{R}^d$. If $\alpha > \nu - 2$ for all $A \in \mathcal{A}_\omega$, then the state $\omega$ is gauge invariant.

**Proof.** If we put $\gamma = \exp \lambda d\Gamma(g), \lambda \in \mathbb{R}, g \in C^\infty_c(\mathbb{R}^d)$ ($C^\infty$ functions in $C_c(\mathbb{R}^d)$) in the Bogoliubov inequality, then it becomes