How Fat is a Fat Bundle?

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Abstract. Let $P \to M$ be a principal $G$-bundle with connection 1-form $\theta$ and curvature $\Theta$. For a subset $S$ of $g^*$ the given connection is $S$-fat (Weinstein, [5]) if for every $\mu$ in $S$ the form $\mu \cdot \Theta$ is nondegenerate on each horizontal subspace in $TP$.

Let $K$ be a compact group and $K/H$ be its coadjoint orbit. The orthogonal projection $\pi : \mathfrak{k} \to \mathfrak{h}$ defines a connection on the principal $H$-bundle $K \to K/H$. We show that this connection is fat off certain walls of Weyl chambers in $\mathfrak{h}^*$. We then apply the result to the construction of symplectic fiber bundles over $K/H$. As an example, we show how higher-dimensional coadjoint orbits fiber symplectically over lower-dimensional orbits.

1. Introduction

Consider a principal $G$-bundle $P \to M$ with connection 1-form $\theta$ and curvature $\Theta$. Recall that $\Theta$ is a horizontal $g$-valued $G$-equivariant 2-form. We obtain horizontal real-valued forms on $P$ by pairing $\theta$ with points in $g^*$.

DEFINITION. The given connection is fat at a point $\mu$ in $g^*$ if the form $\langle \mu, \Theta \rangle$ is nondegenerate on each horizontal subspace in $TP$.

As an immediate consequence of the equivariance of curvature, we see that the connection is then fat at all points of $G \cdot \mu$, the coadjoint orbit through $\mu$. Also, being fat is an open condition. Finally, if the connection is fat at $\mu$ it is fat at any positive multiple of $\mu$, so the collection of points at which the connection is fat at an open $G$-invariant conic set.

In this Letter we shall investigate the fatness of a rather special connection. Let $K/H$ be a coadjoint orbit of the compact group $K$. The Lie algebra of $K$ possesses a $K$-invariant inner product $B$. The orthogonal projection (with respect to $B$) $\theta_B : \mathfrak{k} \to \mathfrak{h}$ gives rise to a left-invariant $\mathfrak{h}$-valued 1-form $\theta$ on $K$. It is a connection on the principal $H$-bundle $K \to K/H$. We shall see that this connection is fat off certain walls of Weyl chambers in $\mathfrak{h}^*$. This result is then applied to fibration of coadjoint orbits.

2. Sternberg's Construction

The motivation for determining explicitly the fatness of the connection comes from symplectic geometry. Specifically, the question arises in Sternberg's construction of a symplectic fiber bundle. More precisely, we have the following theorem.
THEOREM 1. Let $Q$ be a Hamiltonian $G$-space with a moment map $J: Q \to g^*$. Suppose the connection $\theta$ is fat at all points of $J(Q)$. Then the associated bundle $P \times_G Q$ possesses a symplectic structure.

Remark. This theorem is (basically) Weinstein’s Theorem 3.2 in [5]. See also [6]. The idea of using a connection for putting a (pre)symplectic structure $P \times_G Q$ is due to Sternberg [4].

Proof. The connection is fat in some $G$-invariant neighborhood $W$ of $J(Q)$, $W \subseteq g^*$. We first show how to make $P \times W$ into a Hamiltonian $G$-space. Let $pr: P \times W \to W$ denote the projection on the second factor; we can think of $pr$ as a $g^*$-valued function on $P \times W$. Then $\langle pr, \theta \rangle$ is a real-valued 1-form on $P \times W$. Moreover, since $\theta$ is a connection, the form is $G$-invariant. We claim that $d \langle pr, \theta \rangle$ is symplectic.

Indeed, let $(p, w) \in P \times W$. Then $T_{(p, w)}(P \times W) = T_p P \oplus g^*$. $\theta$ defines a splitting of $T_p P: T_p P = H_p \oplus V_p$, and $\theta | V_p: V_p \to g$ is an isomorphism.

$$d \langle pr, \theta \rangle = \langle pr, d\theta \rangle + \langle dpr \wedge \theta \rangle.$$ (Here $\langle dpr \wedge \theta \rangle(X, Y) = \langle dpr(X), \theta(Y) \rangle - \langle dpr(Y), \theta(X) \rangle$.) Under the identification $T_p P \oplus g^* \cong H_p \oplus g \oplus g^*$ the form $\langle dpr \wedge \theta \rangle_{(p, w)}$ is simply the canonical symplectic form on $g \oplus g^*$. On the other hand, $\langle pr, d\theta \rangle_{(p, w)} = \langle w, d\theta_p \rangle$ is non-degenerate on $H_p$, since the connection is fat at $w$. Thus $d \langle pr, \theta \rangle$ is symplectic.

For $\xi \in g$ let $\xi^* = (\xi_p, \xi_w)$ denote the induced vector field on $P \times W$. Since $\langle pr, \theta \rangle$ is $G$-invariant $l_r (\langle pr, \theta \rangle) = 0$. Hence,

$$l_r(\xi^*) d \langle pr, \theta \rangle = -d l_r(\xi^*) \langle pr, \theta \rangle = -d \langle pr, \theta(\xi_p) \rangle = -d \langle pr, \xi \rangle.$$

Thus, the action of $G$ on $P \times W$ is Hamiltonian with the moment map being $-pr$. It follows that the diagonal action of $G$ on $(P \times W) \times Q$ is also Hamiltonian and that zero is the regular value for the moment map $\Phi: (P \times W) \times Q \to g^*$, $\Phi = J - pr$.

$$\Phi^{-1}(0) = \{(p, w, q) \in P \times W \times Q : w = J(q)\}.$$ So $\Phi^{-1}(0)$ can be identified with $P \times Q$. Therefore, the Marsden–Weinstein reduced space $\Phi^{-1}(0)/G$ is diffeomorphic to $P \times_G Q$. This makes $P \times_G Q$ into a symplectic fiber bundle.

We have proved Theorem 1.

3. The Fatness of $K \to K/H$

In this section we determine the fatness of the connection $\theta: TK \to \mathfrak{h}$ defined in the introduction.

Left multiplication allows us to identify the tangent bundle $TK$ with $K \times \mathfrak{t}$, $e: K \times \mathfrak{t} \to TK (a, X) \to dL_a(X)$. Here $L_a$ denotes left multiplication by $a$. Under the identification the connection $\theta$ is given by $\theta(a, X) = \theta_0(X)$, where $\theta_0: \mathfrak{t} \to \mathfrak{h}$ is the orthogonal projection. Also $d\theta((a, X), (a, Y)) = \theta_0([X, Y])$. The horizontal subbundle