ON THE METHOD OF SYMES FOR INTEGRATING SYSTEMS OF THE TODA TYPE

V. GUILLEMIN
Massachusetts Institute of Technology, Cambridge, Mass. 02139, U.S.A.

and

S. STERNBERG
Harvard University, Cambridge, Mass. 02138, U.S.A. and University of Tel Aviv, Tel Aviv, Israel

ABSTRACT. Let $G = KL$ and $g = k + l$ be Lie group and Lie algebra decompositions. This identifies $k^*$ with $l^*$. Any $G$-invariant function, $f$, on $g^*$ induces by restriction a function $f|_{k^*} = l^*$. We prove a formula which says that the integral curve through $\alpha \in k^*$ is obtained as $b(t)\alpha$, where $a(t) = \exp t\xi$ with $\xi = L_f(\alpha)$,

$$a(t) = b(t)c(t)$$

where $(*)$ is the $KL$ decomposition and where $L_f: g^* \to g$ is the Legendre transform. This generalizes a formula of Symes for the generalized Toda lattice.

Let $g$ be a Lie algebra and $k$ and $l$ subalgebras of $g$ with

$$g = k + l.$$  \hspace{1cm} (1)

This gives a corresponding decomposition of the dual spaces.

$$g^* = l^* + k^*$$ \hspace{1cm} (2)

and, in particular, an identification of $k^*$ with $l^*$. Let $O$ be a coadjoint orbit of $L$ (a Lie group whose Lie algebra is $l$) regarded as a submanifold of $k^*$. A function $f$ on $g^*$ restricts to $O$ and hence defines a Hamiltonian system relative to the natural symplectic structure on $O$. The purpose of this note is to explain the method of Symes for integrating this system when $f$ is an invariant function on $g^*$ and we have a global group theoretical decomposition

$$G = LK$$ \hspace{1cm} (3)

corresponding to (1).

Letters in Mathematical Physics 7 (1983) 113 - 115. 0377-9017/83/0072 - 0113 $00.45.
Copyright © 1983 by D. Reidel Publishing Company.
In the case
\[ g = gl(n) \quad k = O(n) \quad l = \{ \text{lower triangular matrices} \} \quad (4) \]
we can identify \( k^0 \) with \( \text{symm} (n) = \{ \text{symmetric matrices} \} \).

It was first observed by Kostant [5] that one can identify the set of Jacobi matrices as a co-
adjoint orbit \( O \) of \( I \) and that the (finite nonperiodic) Toda lattice equations as formulated by
Flaschka [2] and studied by Moser [7] are just the restrictions to \( O \) of the function \( f(A) = \text{tr} A^2 \)
on \( gl(n) \). This led him to a general principal for proving the complete integrability of such equations
and to a systematic generalization of the Toda equations involving arbitrary semisimple Lie groups
and to the detailed solutions of these equations, cf. [5]. Some of these results were also obtained
by Symes [9] and others [1, 8].

More recently, Symes [10] has given a rather explicit method for solving the Hamiltonian system
Corresponding to an arbitrary invariant \( f \) and arbitrary orbit \( O \). His proof in [10] makes use of an
explicit global coordinate chart and, hence, might seem to be restricted to the special choice (4).
We shall show that the method works whenever there is a global decomposition (3). We shall use
the notion of collective motion as introduced in [3]. We briefly recall some of the basic facts in the
theory of collective motion referring the reader to [3] for details and definition:

Suppose we are given a Hamiltonian action of a Lie group \( G \) on a symplectic manifold \( M \) with
moment map \( \Phi: M \to g^* \). A function \( F \) on \( M \) is called collective if it is of the form \( F = f \circ \Phi \), where
\( f: g^* \to \mathbb{R} \) is a smooth function.

The integration of the Hamiltonian system given by such a collective \( F \) proceeds in three steps:

1. For point \( m \in M \) calculate the \( G \) orbit points \( O \) through the point \( \Phi(m) \).
2. Solve the Hamiltonian system on \( O \) given by the function \( f \, |_O \). Let \( \beta(t) \) be the solution
curve with \( \beta(0) = \Phi(m) \).
3. The function \( f \) determines a map \( L_f \) (the Legendre transformation) of \( g^* \to g \). The image
\[ L_f(\beta(t)) = \gamma(t) \] defines a curve in \( g \). The curve \( \gamma \) can be regarded as a time-dependent vector
field on \( G \) and, hence, determines a curve \( a(t) \) in \( G \) with \( a(0) = e \). Then \( a(t)m \) is the trajectory
through \( m \) of the Hamiltonian system given by \( F \).

In case \( f \) is a \( G \)-invariant function, steps (2) and (3) simplify: In step (2) \( f \, |_O \) is a constant so \( \beta(t) \) is
the constant \( \beta(t) = \Phi(m) \). In step (3) \( \gamma(t) \) is a constant so \( a(t) \) is a one-parameter group. Thus,
for invariant \( f \) the solution curve through \( m \) is
\[ (\exp t\xi)m \quad \text{where} \quad \xi = \xi(m) = L_f(\Phi(m)) \quad (5) \]

The cotangent bundle \( T^*G \) may be identified with \( G \times g^* \) using the left invariant identification.
Left multiplication by \( G \) on itself induces a Hamiltonian action on \( T^*G \) given by
\[ a_1(a, \alpha) = (a_1 a, \alpha) \]
and the moment map for this action, \( \Phi: T^*G \to g^* \) is given by \( \Phi_1(a, \alpha) = a \cdot \alpha \), where \( \cdot \) denotes the
coadjoint action. Right inverse multiplication of \( G \) on itself defines a Hamiltonian action on \( T^*G \)
given by \[ a_2(a, \alpha) = (aa^{-1}, a_2 \cdot \alpha) \] and its moment map is given by \( \Phi_2(a, \alpha) = -\alpha \). In particular, a
function \( F \) on \( T^*G \) is left invariant if and only if \( F(a, \alpha) \) does not depend on \( a \), so that we can
write it as \( F(a, \alpha) = f(-\alpha) \) and so is collective for the right action. The function \( F \) is both right and
left invariant if and only if \( f \) is invariant under the coadjoint representation. In that case, the