REDUCTION AND QUANTIZATION FOR SINGULAR MOMENTUM MAPPINGS

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ABSTRACT. When a Hamiltonian action of Lie group on a symplectic manifold has a singular momentum mapping, the reduced manifold may not exist. Nevertheless, we may always construct a Poisson algebra which corresponds to the functions on the reduced manifold in the regular case. The ideas of geometric quantization are extended to Poisson algebras, and it is shown in an example that quantization may be carried out before or after reduction, with isomorphic results.

1. INTRODUCTION

Gauge invariance of a dynamical system leads to constraints in the extended phase space of the system which are given by vanishing of the generators of the gauge transformations. Quantization of such a system can be achieved either by a quantization of the reduced phase space [7], that is the space of orbits of the gauge group which are contained in the constraint set, or by a quantization of the extended phase space and a subsequent imposition of the quantum constraint conditions, i.e., requiring that the physically admissible states should be gauge invariant [3, 9, 11]. In sufficiently regular cases, both approaches are possible and yield equivalent results [5]. However, in many cases of physical interest, e.g., Yang–Mills theory and general relativity, the constraints have quadratic singularities [1], and the reduction process does not lead to a symplectic manifold.

In this note we propose a generalized reduction procedure leading to a reduced Poisson algebra which coincides in the regular case with the Poisson algebra of the reduced phase space. We give a simple example of a singular constraint in which the reduced Poisson algebra is not the Poisson algebra of a symplectic manifold. Nevertheless, the reduced Poisson algebra in our example can be quantized, yielding results equivalent to the results of a quantization of the extended phase space and a subsequent imposition of the quantum constraint condition.

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2. REDUCED POISSON ALGEBRA

Let \((X, \omega)\) be a symplectic manifold. To each smooth function \(f\) on \(X\) there corresponds a vector field \(\xi_f\), called the Hamiltonian vector field of \(f\), defined by \(\xi_f \cdot \omega = -df\), where \(\cdot\) denotes the left interior product. The Poisson bracket \([f_1, f_2]\) of two smooth functions on \(X\) given by

\[
[f_1, f_2] = -\xi_{f_1} f_2 = \xi_{f_2} f_1 = -\omega(\xi_{f_1}, \xi_{f_2})
\]

defines a Lie algebra structure on \(C^\infty(X)\) satisfying the Leibniz identity

\[
[f_1, f_2 + f_3] = [f_1, f_2] + [f_1, f_3].
\]

In general, an associative commutative algebra with a bracket operation \([,]\) satisfying the Leibniz identity will be called a Poisson algebra. (See [12], where the term canonical algebra is used.)

Let \(G\) be a connected Lie group, \(g\) the Lie algebra of \(G\), and \(g^*\) the dual of \(g\). We assume that there is a Hamiltonian action of \(G\) on \((X, \omega)\) with \(G\)-equivariant momentum map \(J: X \to g^*\) [6, 10]. The function \(J_a\) on \(X\) defined by \(J_a(x) = \langle J(x), a \rangle\), where \(\langle , \rangle\) denotes the evaluation map is called the momentum corresponding to \(a \in g\).

We interpret \((X, \omega)\) as the extended phase space of a dynamical system, with gauge group \(G\) and the constraints given by \(J = 0\). Situations of this type appear in the dynamics of Yang-Mills fields, where \(G\) is the group of gauge transformation [1].

Let \(\mathcal{J}\) denote the ideal in \(C^\infty(X)\) (relative to the associative algebra structure) generated by the momenta \(J_a, a \in g\). From the \(G\)-equivariance of the momentum map \(J\), one has the following lemma.

**Lemma 1.** \(\mathcal{J}\) is a Poisson subalgebra of \(C^\infty(X)\) stable under the action of \(G\).

The action of \(G\) on \(C^\infty(X)\) induces an action of \(G\) on the quotient algebra \(C^\infty(X)/\mathcal{J}\) such that the projection homomorphism \(\rho: C^\infty(X) \to C^\infty(X)/\mathcal{J}\) is \(G\)-equivariant. We denote by \(\mathcal{A}\) the space of \(G\)-invariant elements of \(C^\infty(X)/\mathcal{J}\).

**Lemma 2.** \(\rho^{-1}(\mathcal{A})\) is the normalizer of \(\mathcal{J}\) in the Lie algebra sense, and it has the structure of a Poisson subalgebra of the Poisson algebra \(C^\infty(X)\).

**Proof.** A function \(f \in C^\infty(X)\) is contained in \(\rho^{-1}(\mathcal{A})\) if and only if \(gf = f\) for every \(g \in G\). Since \(G\) is connected this condition is equivalent to \([J_a, f] \in \mathcal{J}\) for every \(a \in g\). But \(\mathcal{J}\) is generated by \(J_a\)'s so that \(f \in \rho^{-1}(\mathcal{A})\) if and only if \([J_a, f] \in \mathcal{J}\) for every \(J_a \in \mathcal{J}\). Thus, \(\rho^{-1}(\mathcal{A})\) is the normalizer of \(\mathcal{J}\).

If \(f_1, f_2 \in \rho^{-1}(\mathcal{A})\), then, for each \(J_a \in \mathcal{J}\),

\[
[J_a, [f_1, f_2]] = [[J_a, f_1], f_2] = [[f_2], f_1] \in \mathcal{J}.
\]

Thus, \(\rho^{-1}(\mathcal{A})\) is a Poisson subalgebra of the Poisson algebra \(C^\infty(X)\).