MULTI-SOLITON SOLUTIONS TO THE THIRRING MODEL THROUGH THE REDUCTION METHOD*

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RESUMÉ. It est démontré que la méthode de réduction appliquée à un système d'ordre 2 × 2
du type Zakharov-Shabat, muni d'une structure méromorphe appropriée, amène au modèle
classique de Thirring; ce dernier étant de fait la condition d'intégrabilité du précédent système.
Réduisant les transformations de Bäcklund génératrices de solitons, les solutions multi-solitons
sont dérivées de manière explicite.

ABSTRACT. It is shown how the reduction method applied to a 2 × 2 Zakharov-Shabat system
with appropriate meromorphic structure leads to the Thirring model as integrability conditions.
Reducing the generic soliton-generating multi-Bäcklund transformations, the general multi-soliton
solutions are explicitly derived.

1. INTRODUCTION

The Thirring model:

\[ \begin{align*}
\phi_{1 \eta} &= 2ig^2 1 \phi_2 l^2 \phi_1 - im\phi_2, \\
\phi_{2 \xi} &= 2ig^2 1 \phi_1 l^2 \phi_2 - im\phi_1,
\end{align*} \]

(1.1)

\[ \phi_1 (\xi, \eta), \phi_2 (\xi, \eta) \in \mathbb{C} \]

has been studied as an example of an integrable 1 + 1-dimensional relativistic system by several
authors [1–3]. Kuznetsov and Mikhailov [1, 2] have worked out the inverse scattering trans-
form which allows, in principle, the determination of all localized solutions. In particular, they
have explicitly calculated the one-soliton solution and the phase shifts for soliton collisions. As
is usual in such systems, the full inverse scattering method, which requires the solution of a
Gelfand-Levitan-Marchenko type integral equation, is more cumbersome than necessary if
one is only interested in obtaining explicit multi-soliton solutions. For the latter, simpler

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approaches, such as Bäcklund transformations, lead directly to the results.

In this letter we show how application of the reduction procedure, as formulated by Mikhailov [4], and further developed by two of the present authors [5], gives rise very simply to explicit multi-soliton solutions to the Thirring model. The general framework for applying the reduction method to multi-Bäcklund transformations and their solution is fully detailed in [5].

2. THE THIRRING MODEL AND REDUCTION PROCEDURE

We begin with the 2 × 2 Zakharov–Shabat system:

\[
\psi_\xi = U(\lambda)\psi, \quad \psi_\eta = V(\lambda)\psi
\] (2.1)

where

\[
U(\lambda) = U_0 + U_1\lambda + U_2\lambda^2, \quad V(\lambda) = V_0 + V_1\lambda^{-1} + V_2\lambda^{-2}
\] (2.2)

and \(U_0, U_1, U_2, V_0, V_1, V_2\) are traceless matrix functions of the light-cone variables \((\xi, \eta) \in \mathbb{R}^2\) and \(\psi(\lambda, \xi, \eta)\) is an \(\text{SL}(2, \mathbb{C})\)-valued function depending on the complex parameter \(\lambda\). The integrability conditions for (2.1) are:

\[
U_{0,\eta} - V_{0,\xi} + [U_0, V_0] + [U_1, V_1] + [U_2, V_2] = 0,
\]

\[
U_{1,\eta} + [U_1, V_0] + [U_2, V_1] = 0, \quad U_{2,\eta} + [U_2, V_0] = 0,
\] (2.3)

\[
V_{1,\xi} + [V_1, U_0] + [V_2, U_1] = 0, \quad V_{2,\xi} + [V_2, U_0] = 0.
\]

Applying the reduction procedure [4, 5], we impose the following conditions, which amount to invariance under an order-four automorphism group of the system (2.1):

\[
\psi^+(\lambda) = \psi^{-1}(e\lambda),
\] (2.3a)

\[
\tau \psi(\lambda)\tau^{-1} = \psi(-\lambda)
\] (2.3b)

where

\[
\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \pm 1.
\] (2.4)

The infinitesimal form of the invariance conditions gives, by Equation (2.1):

\[
U^+(\lambda) = -U(e\lambda), \quad V^+(\lambda) = -V(e\overline{\lambda})
\] (2.5a)