EIGENFUNCTION ASYMPTOTICS FOR SYSTEMS OF THREE CHARGED PARTICLES

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ABSTRACT. The coordinate asymptotics of eigenfunctions is investigated for a system of three particles with a Coulomb interaction. The confluent hypergeometric function is used for describing the asymptotics in some directions of configuration space.

1. INTRODUCTION

Asymptotic form is known to be an important property of eigenfunctions used for studying the discrete spectrum of quantum mechanical Hamiltonians (see, for example [1-6]). For systems with short-range potentials, the exact asymptotic form of the three-particle eigenfunctions was investigated in [6]. For three Coulomb particles, however, only estimates are available [1-4].

In this paper, we shall describe the asymptotic form of the three-body eigenfunctions for a system of charged particles. We shall establish that these functions have a structure analogous to the one founded for neutral particles. In particular, the decreasing exponents are the same as for the neutral particles. However, new power factors appear which make the eigenfunctions decrease faster or slower due to repulsive or attractive Coulomb forces. Also, a new special function is necessary to describe the asymptotics of the eigenfunctions in the domains where the latter changes its form; namely, the function erfc(t) is replaced by the irregular hypergeometric function Ψ(a, c, t) [7].

To investigate the asymptotic form of the eigenfunctions we apply the eikonal approximation method used in [8] for studying the scattering wavefunctions. In this method, one constructs a formal solution of the Schrödinger equation corresponding to an eikonal \( L, l \text{\,\,} V_L \text{\,\,} = \text{\,\,} 1 \), as a series in powers of energy \( E^{-k/2} \) and define the coefficients by recurrence relations [8]. The rigorous proof of so-derived formulas may be obtained by using modified configuration space Faddeev equations [8].

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2. NOTATION

We consider a system of three nonrelativistic pairwise interacting charged particles. The pair interacting potentials $V_{\alpha}(x)$ will be assumed to have the form of the sum of the Coulomb part and a nuclear short-range one,

$$V_{\alpha}(x) = n_{\alpha} |x|^{-1} + V_{\alpha}^{n}(x), \quad \alpha = 1, 2, 3.$$ 

We suppose the two-particle Hamiltonians $h_{\alpha} = -\Delta + V_{\alpha}(x)$ to have the bound states $\Psi_{\alpha,i}(x) (\alpha = 1, 2, 3; i = 1, 2, ...)$ with negative bound energies $-\kappa_{\alpha,i}^2$. We shall denote by $\Psi_{\alpha}(x)$ and $-\kappa_{\alpha}^2$ the ground-state eigenfunctions and eigenvalues respectively, $\kappa_{\alpha}^2 = \max_{i} \kappa_{\alpha,i}^2$.

We take as the independent coordinates in the three-body center of the mass system the usual relative coordinates $x_{\alpha}, y_{\alpha} (\alpha = 1, 2, 3)$ which are combined into the six vectors $X = \{x_{\alpha}, y_{\alpha}\}$, $X^2 = x_{\alpha}^2 + y_{\alpha}^2$, of the configuration space $R^6$ [6]. The different pairs are linearly related by

$$x_{\beta} = c_{\beta\alpha} x_{\alpha} + s_{\beta\alpha} y_{\alpha}, \quad y_{\beta} = -s_{\beta\alpha} x_{\alpha} + c_{\beta\alpha} y_{\alpha}, \quad c_{\beta\alpha}^2 + s_{\beta\alpha}^2 = 1,$$

where the coefficients $c_{\beta\alpha}$ and $s_{\beta\alpha}$ are expressed in terms of particle masses [6].

The three-body Hamiltonian $H$ is given by the relation

$$Hf = \left(-\Delta + \sum_{\alpha} V_{\alpha}(x_{\alpha})\right)f(X),$$

where the Laplacian in $R^6$ corresponds to the free Hamiltonian $H_0$. For any pair $\alpha = 1, 2, 3$ we have the representation $\Delta X = c_{\alpha}\Delta x_{\alpha} + s_{\alpha}\Delta y_{\alpha}$. We denote by $-E_i (i = 1, 2, ...)$, $E_i > \max_{\alpha} \kappa_{\alpha}^2$, the eigenvalues of $H$ lying outside the continuous spectrum of $H$ and the corresponding bound states by $\Psi_i(X)$. Below, we omit the label $i$.

3. EIKONAL APPROXIMATIONS

Let $L$ be a solution of the eikonal equation $|\nabla L| = 1$. The representation

$$X = L \nabla L + M, \quad (M, \nabla L) = 0$$

defines an orthogonal coordinate system in $R^6$ due to the eikonal $L$. We call the $L$-eikonal approximation $\Psi_L$ the expression

$$\Psi_L = A_L f_L(M) W_L \exp\{-\sqrt{EL}\}, \quad (1)$$

where $A_L$ is a solution of the continuity equation $2(\nabla L, \nabla A) + A \nabla L = 0$. The eikonal factor $W_L$ being the phase shift along the trajectory $j(t) = t \nabla L + M$

$$W_L = \exp\{- (2\sqrt{E})^{-1} \int_{\mathcal{H}} V_{\alpha}(t \nabla L + M) \ dt\}.$$