ABSTRACT. Wong [14] introduced equations of motion for a spin 0 particle in a Yang-Mills field which was widely accepted among physicists. It is shown that these are equivalent to the various mathematical formulations for the motion of such particles as given by the Kaluza-Klein formulation of Kemer [4], and those of Sternberg [11], and Weinstein [12]. In doing this, we show that Sternberg's space is, in a natural way, a symplectic leaf of a reduced Poisson manifold and relations to a construction of Kummer's [5] for dynamics on the cotangent bundle of a principle bundle are clarified.

1. INTRODUCTION

Wong's equations are

\[ \dot{\xi}_a = \frac{1}{2} \frac{\partial g^{a\nu}}{\partial x^\mu} \dot{\xi}_a \dot{x}^\mu - \frac{1}{2} \frac{\partial g^{a\nu}}{\partial x^\mu} \dot{x}^\mu \xi_a, \]  

\[ \dot{x}^\mu = -c^d_{ab} A^{ab}_\mu \dot{x}^d, \]

where \( g \) is a Lorentz-metric on a space-time \( X \), \( A \) is a connection on a principal bundle \( P \) over \( X \), \( F \) is its curvature, \( \xi_a \) are a basis for the Lie algebra \( \mathfrak{g} \) of the structure group \( G \) of \( P \), the \( c \)'s are the structure constants of \( \mathfrak{g} \), \( x^\mu \) are space-time coordinates, and \( \dot{x}^\mu \) are the resulting momentum (cotangent) coordinates. Space-time indices are raised and lowered by \( g \). We will assume there is a bi-invariant metric \( \gamma \) on \( \mathfrak{g} \) with which Lie algebra indices can be raised and lowered. These equations are amended by

\[ \dot{x}^\mu = p^\mu, \]

and the interpretation \( \dot{x} = \frac{m}{\gamma} (\dot{r} \gamma), \) where \( m \) is the particle's rest mass, and \( \tau \) its proper time. For convenience the coupling constant and Planck's constant have been set equal to one.

The geometric interpretation of these equations is as follows. Equation (1a) is that for the worldline of a particle under a generalized Lorentz force. Equation (1b) says that the isotropic spin momentum, which is a section of the associated co-adjoint bundle.
\[ E = P \times_G \mathfrak{g}^* \text{ (with fibre coordinates } \xi_a) \]

over the wordline, is parallel translated by the connection induced on \( E \) by \( A \).

There are three other formulations of the dynamics of a particle in a Yang-Mills field. The Kaluza-Klein picture of electromagnetism was given a straightforward generalization to the Yang-Mills case by Kerner \([4]\). Sternberg \([11]\) gave the first formulation in the spirit of the modern school of symplectic geometers. Lastly, soon after Sternberg, Weinstein \([12]\) gave a natural formulation using reduction and showed it was equivalent to Sternberg’s. Sniatycki \([10]\) showed that Kerner’s and Weinstein’s, hence Sternberg’s, formulations predict the same worldlines on space-time. A straightforward calculation done in Section 2 shows that this worldline is also given by Wong’s equation \((1a)\).

It has been unclear, however, how the various formulations are related in the fibre direction, that is, to Wong’s equation \((1b)\). In this paper we show, that all formulations are equivalent in a natural way (Theorems 1 and 2).

A guiding principle for this work has been that Wong’s equations naturally live on the vector bundle \( E^\# = P^\# \times_G \mathfrak{g}^* \) and not on the submanifold of \( E^\# \) given by \( \mathcal{O}^\# = P^\# \times_G \mathcal{O} \), which is Sternberg’s phase space. Here \( P^\# \) denotes the pullback of \( P \) obtained by completing the following diagram with dotted arrows

\[
\begin{array}{ccc}
P^\# & \longrightarrow & P \\
\downarrow & & \downarrow \\
T^*X & \longrightarrow & X
\end{array}
\]

and \( \mathcal{O} \) is a co-adjoint orbit in \( \mathfrak{g}^* \). The key to the equivalence of the various formulations was the realization that \( A \) induces a Poisson structure on \( E^\# \) whose symplectic leaves are the \( \mathcal{O}^\# \)’s (see Theorem 2).

2. KALUZA-KLEIN AND KERNER

In the Kaluza-Klein picture of the motion of a colored particle in a Yang-Mills field, as generalized by Kerner, the motion of the particle is a geodesic on \( P \) where the metric \( K \) on \( P \) is induced by the connection \( A \). That is,

\[ T_p^0P = V_p \oplus H_p, \quad p \in P \quad (2) \]

is an orthogonal decomposition, where \( V_p \) is the vertical space and \( H_p \) is the connection’s horizontal distribution. On \( V_p, K \) is induced by the fixed bi-invariant metric \( \gamma \) on \( \mathfrak{g} \) under the infinitesimal generator isomorphism

\[ \sigma_p: \mathfrak{g} \to V_p \subseteq T_pP \quad (3) \]

and on \( H_p \) it is the horizontal lift of the metric \( g \) on space-time, via the connection.