NONLINEAR EQUATIONS WITH SUPERPOSITION PRINCIPLES
AND THE THEORY OF TRANSITIVE PRIMITIVE LIE ALGEBRAS

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ABSTRACT. The classification of systems of nonlinear ordinary differential equations with superposition principles is reduced to a classification of transitive primitive Lie algebras. Each system can be associated with the transitive primitive action of a Lie group $G$ on a homogeneous space $G/H$, where $H$ is a maximal subgroup of $G$. The equations can have specific polynomial or rational nonlinearities.

1. INTRODUCTION

The purpose of this article is (1) To reduce the problem of classifying all systems of nonlinear ordinary differential equations with superposition principles to that of classifying primitive transitive filtered Lie algebras. (2) To apply the existing classification of such Lie algebras to construct typical examples of ODE's with superposition principles explicitly. We restrict our attention to systems of first-order ordinary differential equations

$$\frac{dx^\mu}{dt} = f^\mu(x, t), \quad \mu = 1, ..., n, \quad x, f \in \mathbb{R}^n. \quad (1.1)$$

We say that Equations (1.1) admit a superposition formula if there exists a finite number $m$ of 'generic' solutions $x_1(t), ..., x_m(t)$ and a function

$$S: \mathbb{R}^{m(m+1)} \rightarrow \mathbb{R}^n \quad (1.2)$$

such that the general solution to (1.1) can be written as

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\[ x(t) = S(x_1(t), \ldots, x_m(t), c), \quad (1.3) \]

where \( c \) is a constant vector, related to the initial conditions for \( x(t) \).

From this point of view, linear equations are just a special case: the superposition formula (1.3) is then a linear combination of \( m = n \) linearly independent solutions. More generally, Lie has shown [1] that system (1.1) allows a superposition formula in the above sense, if and only if the equations can be written in the form

\[
\frac{dx^\mu}{dt} = \sum_{i=1}^{k} a_i(t) \xi^\mu_i(x), \quad 1 \leq \mu \leq n \quad (1.4)
\]

and the vector fields defined as

\[
\tilde{\xi}_i(x) = \sum_{\nu=1}^{n} \xi^\nu_i(x) \frac{\partial}{\partial x^\nu} \quad (1.5)
\]

generate a finite-dimensional subalgebra of the algebra of vector fields on \( \mathbb{R}^n \). The problem of classifying systems of \( n \) ODE's with superposition formulas is thus reduced to that of classifying finite-dimensional Lie algebras that can be realized in terms of vector fields in \( n \) variables. This in itself is a formidable task, even for such low values of \( n = 2 \). Lie himself, in a different context [2], performed this classification for vector fields on \( \mathbb{R}^1 \) and \( \mathbb{R}^2 \).

For \( n = 1 \) the only algebras that can be realized are \( \text{sl}(2, \mathbb{R}) \) and its subalgebras, leading, via relations (1.4) and (1.5), to the Riccati equation \( \dot{y} = a_1(t) + a_2(t)y + a_3(t)y^2 \) (or to linear equations). For \( n = 2 \) infinitely many mutually nonisomorphic Lie algebras (of arbitrary dimensions) can be realized. Three of them, \( \text{sl}(3, \mathbb{R}) \), \( \text{so}(3, 1) \), and \( \text{so}(2, 2) \) correspond in modern terms, to two-dimensional symmetric spaces and lead to coupled Riccati equations. The remaining algebras (that are not subalgebras of the above three) contain nontrivial solvable ideals. The system of two equations turns out to be decomposable: a Riccati or linear equation is solved in one variable. Once this equation is solved and the solution substituted into the other equation, this one also reduces to a linear or Riccati equation. The desire to generalize this 'decoupling' result to arbitrary values of \( n \) leads to a consideration of primitive transitive group actions on \( \mathbb{R}^n \).

Earlier studies have been devoted to superposition principles for systems of projective, conformal and matrix Riccati equations [3-5]. In each of these cases the equations and the superposition formulas can be interpreted in the following manner. Consider the transitive action of a Lie group \( G \) on a homogeneous space \( G/H \) where \( H \subset G \) is a Lie subgroup of \( G \). Construct the vector fields \( \tilde{\xi}_i(x) \), corresponding to the infinitesimal action of \( G \) on \( G/H \) in the neighbourhood of the origin \( x_0 \) (in some appropriately chosen coordinates). The vector fields \( \tilde{\xi}_i(x) \) generate the Lie algebra \( L \) of the Lie group \( G \), the subalgebra \( L_0(x_0) \) of vector fields vanishing at the origin \( x_0 \) corresponds to the isotropy group \( H \) of the origin. Use relations (1.4) and (1.5) to associate a system of ODE's (1.4) to the algebra \( L \).

The solution \( x(t) \) of (1.4) is then given as

\[
x(t) = g(t)x_0, \quad x_0 = c, \quad (1.6)
\]