On the Interpretation of Anticommuting Variables in the Theory of Superspace

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Abstract. On the basis of cohomology theory, we consider a possible topological interpretation of Grassmann algebra used in the definition of superspace.

1. Introduction

A rigorous mathematical theory of superspace (supermanifold) was elaborated in considerable detail by several authors (F. A. Berezin, D. A. Leites, B. De Witt, A. Rodgers, and others). In this Letter, we follow the terminology and definitions of Rodgers [1] in that respect (although the vector spaces and algebras considered here are defined over the complex field C and not over the reals, as in [1]). Accordingly, let us denote $B_L$ as a complex Grassmann algebra over $C_L$, where $L$ is a finite integer (we consider only this finite-dimensional case below). A basis in $B_L$ consists of a unity 1, of elements $\beta_i$ ($i = 1, \ldots, L$) satisfying $\beta_i \beta_j + \beta_j \beta_i = 0$ (for all $i, j$) and of products $\beta_i \ldots \beta_k$ (where $1 \leq i_1 < i_2 < \cdots < i_k \leq L$, $2 \leq k \leq L$). As is known, an essential feature of $B_L$ from the point of view of the theory of superspace is $B_L$ being a $Z_2$-graded algebra, $B_L = B_{L, 0} + B_{L, 1}$, where $B_{L, 0}$ is a subalgebra of even elements and $B_{L, 1}$ is a subspace of odd elements. We do not consider supermanifolds in this Letter, but limit ourselves to superspaces of the type $B_{m, 0} \times B_{n, 1} \equiv B_{m, n}$. Elements of a superspace are denoted $(\zeta_1, \ldots, \zeta_m, \theta_1, \ldots, \theta_n) \equiv (\zeta, \theta)$; further,

$$
\zeta_k = z_{k, 0} + 1 + \sum_{(e)} z_{k, (e)} \beta_{(e)},
\quad \theta_l = \sum_{(u)} z_{l, (u)} \beta_{(u)}.
$$

Here $k = 1, \ldots, m; l = 1, \ldots, n; (e)$ is an index labelling even elements of the basis of an algebra $B_L$ and $(u)$ labels odd elements of the $B_L$'s basis. $z_{k, 0}, z_{k, (e)}, z_{k, (u)}$ are complex numbers. Let us remark that the definition of the coordinates $\zeta$ and $\theta$ (by $m = n = 1$ and an arbitrary $L$) is basic in the theory of super-Riemann surfaces applied for investigation of the superstring (see, e.g., [2]). In the following, however, we do not have in view any specific physical interpretation and also do not specify any particular groups of super-automorphisms. All that seems interesting for us in this Letter in the first place, is the possibility of a topological interpretation of odd superspace coordinates.
2. A Connection with the Cohomology Theory

In common formulations of the theory of superspace, a Grassman algebra $B_L$ essentially plays an auxiliary rôle, its dimensionality being arbitrary except that it must be big enough so that both even variables $\zeta_k$ and odd ones $\theta_i$ are linear independent. However, the possibility seems very attractive to connect the Grassmann algebra $B_L$, by using some well-known mathematical results, with a cohomology ring of an auxiliary topological space for which this ring has the structure of a Grassmann algebra (the simplest example of such a space is a torus, see below; but this also has place for compact Lie groups of a more general type). To expound the idea of such a comparison, it is sufficient to consider the superspace $B_L^{1,1}$ of the simplest type corresponding to a superstring. It is useful to interpret this superspace as a trivial vector bundle with the base $B_{L,0}$ and the fibre $B_{L,0}$. Note that the fibre (the base) of this trivial vector bundle is a vector group consisting of all even (odd) elements of the Grassmann algebra.

Consider now an auxiliary compact complex manifold $K$ with a Hermitian metric on it and complex-valued differential forms of the type $(p, q)$ on the manifold $K$, i.e., sections of the vector bundle $\Lambda^{p,q}T^*(K)$. The first of the necessary mathematical results mentioned above consists of the existence of an isomorphism $\mathcal{H}^{p,q}(K) \cong H^{p,q}(K)$, where $\mathcal{H}^{p,q}(K)$ is the space of harmonic forms of the type $(p, q)$ and $H^{p,q}(K)$ is the Dolbeault cohomology group of the manifold $K$. In addition, Hodge's theorem states that the space $\mathcal{H}^{p,q}(K)$ is finite-dimensional.

Now, for developing a connection of interest, it is necessary to recall some results of the cohomology theory which assume an elementary character in the case when $K$ is a complex torus [3]. Let $\text{Cl}(\alpha)$ be the cohomology class of some closed form $\alpha$ defined on an (arbitrary) compact complex manifold. If $a = \text{Cl}(\alpha)$, $b = \text{Cl}(\beta)$, define $a \wedge b = \text{Cl}(\alpha \wedge \beta)$. Just this (Alexander-Kolmogoroff) composition law transforms the set of cohomology classes into an algebra with a multiplication of the outer-product type. Further, an operator $H$ can be defined possessing a property that, for each closed form $\alpha$ on the manifold, $H\alpha$ is a harmonic form homologic to $\alpha$. In addition, if $\alpha \sim \alpha'$, then $H\alpha = H\alpha' \equiv Ha$. However, $\alpha$ and $\beta$ being harmonic forms, $\alpha \wedge \beta$ in general is not necessarily harmonic. Yet if $K$ possesses more specific properties (for example, if $K$ is a torus), then the mapping $a \mapsto H\alpha$ is not only a vector space isomorphism between cohomology classes and harmonic forms, but also an isomorphism between algebras with the composition law $\wedge$. Let a torus have a real dimensionality $t$ (assuming also that a translation invariant form $dx^2$ is defined on it). It can be identified with a factor-space $V/D$ of a Euclidean space $V$ of dimension $t$ with an inner product defined on it by a rank $t$ discrete subgroup $D$ of this space. If $f$ is a function of the torus, then the operator $H$ is defined so that [3]

$$Hf = a(0) = \mu_{V/D}[f],$$

where $a(0)$ is a coefficient of unity in a Fourier series expansion of the function $f$ and $\mu_{V/D}$ is an average of the $f$ on the torus relative to the invariant measure $dx_1 \ldots dx_t$. Finally,

$$H(f \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = (Hf) \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$