SUPERSELECTION VARIABLES AND GENERALIZED MULTIPLIERS

N. GIOVANNINI
Département de Physique Théorique, Université de Genève, 1211 Genève 4,
Switzerland

ABSTRACT. We discuss and solve the problem of phase factor families that arises in the representation theory of the symmetries of an elementary physical system, the latter being expressed in terms of its propositional lattice.

0. INTRODUCTION

In a previous paper [1] we have shown that we could derive both the quantal and the classical state spaces for a spinless particle, both in the relativistic and in the non-relativistic contexts, within a common group theoretical framework involving representations on direct unions of Hilbert spaces. Beside the two extreme cases where the union is trivial (the purely quantal case) or where the Hilbert spaces are all one-dimensional (the purely classical case), our framework allows us to treat the intermediate cases where some observables are of the classical type and play the role of, possibly continuous, superselection parameters.

To extend these models, in particular if we want to consider particles with arbitrary spin, or for the discussion of the non-relativistic limit of the relativistic model we need, however, to treat the more general case with arbitrary projective representations. For that purpose we first need to determine the corresponding possible families of phase factors. This is the subject of the present paper where we give the general solution of this problem.

1. K-SPACES AND PROJECTIVE K-REPRESENTATIONS

Let $K$ be a direct union over some Borel index set $S$ of a family of isomorphic Hilbert spaces (that will be assumed here to be complex and separable). On $K$ there exists a natural lattice of propositions defined by the projections in $K$, i.e., by the families $\{P_s\}$ of projectors in the corresponding Hilbert spaces. From the lattice structure of $\mathcal{P}(\mathcal{H}_s)$, the set of all projectors in $\mathcal{H}_s$, one obtains (cf., [2]) a natural lattice structure $\mathcal{L}(K)$ on the projections in $K$:

$$\mathcal{L}(K) = \bigvee_{s \in S} \mathcal{P}(\mathcal{H}_s).$$

(1.1)

It now follows from the representation theorem of Piron [2] that (up to some technical restrictions) the above $K$-space describes the most general allowable state space of a physical system.
which is expressed in terms of a lattice of proposition isomorphic to \( \mathcal{L}(K) \). The algebraic structure of \( \mathcal{L}(K) \) allows us to apply the notion of morphism and, correspondingly, a symmetry of \( K \) is defined as being an automorphism of the corresponding lattice of propositions. The essential result that we shall need in the sequel is the following generalization of the theorem of Wigner.

**Theorem 1.1** [3]. *Every symmetry of a proposition system defined by a family \( \{ \mathcal{H}_s, s \in S \} \) is given by a permutation \( \pi \) of the index set \( S \) and a family of unitary or antiunitary transformations \( U_{\pi(s)}: \mathcal{H}_s \rightarrow \mathcal{H}_{\pi(s)} \). Moreover each \( U_s \) is unique up to a phase.*

As a consequence of this theorem, and the symmetries forming a group \( G \), one has thus, \( \forall g_1, g_2 \in G, s \in S \)

\[
U_s(g_1)U_{\pi^{-1}(s)}(g_2) = \omega_{s}(g_1, g_2)U_s(g_1g_2)
\]

(1.2)

where \( \omega_{s}(g_1, g_2) \in U(1) \) is a phase that may depend on \( g_1, g_2 \) and \( s \). These phases are just the objects that we are going to study in detail in the present paper. But first, using the above theorem, we generalize the usual notion of representation as follows:

**Definition 1.2.** A **projective \( K \)-representation** of \( G \) is a set of automorphisms \( U(g): K \rightarrow K \), \( \forall g \in G \) that satisfies the following four conditions

(i) \( U(e) = \mathbb{1}_K \), \( e \) the unit of \( G \) (1.3)

(ii) \( U(g_1)U(g_2) = \Omega(g_1, g_2)U(g_1g_2) \) (1.4)

with \( \Omega \) a family \( \{ \omega_s \} \) of \( c \)-numbers of length 1 whose action on an element \( \Psi \) of \( K \) is defined by ordinary multiplication, in each Hilbert space.

(iii) \( (U(g)\Psi)_s = L_{x(s)}\psi_{\pi^{-1}(s)}(\Psi)^{\pi^{-1}(s)} \) (1.5)

with \( L_s \in \mathfrak{H}(\mathcal{H}_s) \), the group of all unitary/antiunitary operators in \( \mathcal{H}_s \), \( \mathcal{H}_s \), the imbedding map of \( \mathcal{H}_s \) in \( \mathcal{H}_{\pi(s)} \), and where we have, by a slight abuse of notation, denoted \( gs \) the image \( \pi_g(s) \) of \( s \) with \( \pi_g \) the permutation induced on \( S \) by \( U(g) \).

(iv) \( U(g) \) is continuous in \( g \) on \( S \times \mathcal{H} \) (with for \( \mathcal{H} \) the strong topology).

Up to some convenient changes in notation, this is the natural projective extension of the definition given in [1] and, as in [1] \( S \) becomes a \( G \)-space in the sense of Mackey [4]. In particular, if the action of \( G \) is transitive on \( S \), then \( S \) can be identified with a quotient space \( G/H \) with \( H = \text{Stab } s_0 \), the stabilizer of an arbitrary point \( s_0 \in S \).

**Definition 1.3.** A **projective \( K \)-representation** is called irreducible if and only if