SOLITON-SHAPED POTENTIALS AND THE GENERALIZED POWER-SERIES METHOD

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ABSTRACT. With the double-well potential appearing in the soliton theory, the one-dimensional Schrödinger equation is solved exactly in terms of the quasihypergeometric series. The method is applicable to the arbitrary potentials of the form

$$V_q(r) = \sum_{n=0}^{\infty} g_n/(\alpha + \chi r)^n.$$ 

1. INTRODUCTION

The perturbed Sine–Gordon equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \phi_k(\phi), \quad \phi_k(\phi) = \sin \phi + k \sin \frac{\phi}{2}, \quad k \leq 1 \quad (1.1)$$

introduces, in a natural way, the binding of the Sine–Gordon soliton-soliton pairs [1]. Their perturbative description may be based on the exact static ground-state formula \( \phi = U_k = 4 \arcsin (\chi r + \alpha)^{-1/2}, \alpha > 0, r = \text{const.} x \) [2]. In the lowest order, we put \( \phi = U_k + \Delta \phi, \Delta \phi = \sum_n \psi_n \exp (i\omega_n t) \) and get from (1.1) the linearized Schrödinger-like equation

$$- \frac{d^2}{dr^2} \psi(r) + V_q(r)\psi(r) = E \psi(r), \quad q = 2, \quad V_2(r) = -\frac{g}{\alpha + \chi r} - \frac{h}{(\alpha + \chi r)^2} \quad (1.2)$$

for the frequencies \( \omega_n^2 = E + \text{const.} \) and the Fourier components \( \psi_n = \psi(r) \) of the excited-state correction \( \Delta \phi \). It is worth mentioning also that the other deformations

\( (\phi_k(\phi) = \text{sn} (\phi/2, k) \text{cn} (\phi/2, k)/\text{dn}^3 (\phi/2, k) [2]) \) of the Sine–Gordon interaction term \( \phi_0 \) lead to the same form (1.2). This equation itself finds further application in the discussion of the stability or within the quantization formalism for solitons [3]. Even in the old quantum-mechanical Schrödinger equation interpretation, the potential well of the type \( V_\chi(r) \) with the 'core' or two 'barriers' may be of interest in the molecular physics, etc.

We shall present here the compact and exact power-series solution of (1.2) which may easily be extended to any order \( q \) of the polynomial \( V_q(r) = f(1/\alpha + \chi r) \). In brief, we arrange both the powers of the independent variable and the coefficients into the \( q \)-dimensional vectors and define the generalized power series by means of their scalar products. In detail, we shall consider the bound-state solutions only.
2. THE HYPERGEOMETRIC EXAMPLE \((q = 1)\)

From the formal point of view, Equation (1.2) with the special \(\alpha = 1, \hbar = 0\) case of \(V_2(r)\) (the 'modified Pöschl-Teller' \([4]\) or 'Sine—Gordon-shaped' \([2]\) potential) coincides with the Gauss hypergeometric equation. On this example with \(V_q(r) = -\lambda(\lambda - 1)/2(1 + \chi r), E = \kappa^2, q = 1\) and the \(p = 0\) or \(p = 1\) fundamental solutions

\[
\psi(r) = \text{const. th}^p \frac{r}{2} \text{ch}^{-2\kappa} \frac{r}{2} {}_2F_1 \left( \kappa + \frac{p}{2}, \kappa + \frac{p}{2} + \frac{1 - \lambda}{2}, 1 + 2\kappa, \frac{1}{\text{ch}^2(r/2)} \right)
\]

we shall illustrate the overall structure of the method and compare the three methodical alternatives of Morse and Feshbach \([5]\) (A), Flugge \([4]\) (B), and the present one (C), respectively.

(A) The variable transformation \(r \rightarrow z = (1 + \text{th}(r/2))/2\) in (1.2) implies the convergence of the hypergeometric series solution \(z^F_1(\cdots, z)\) in the full region \(r \in (-\infty, \infty)\). When the asymptotically wrong \(e^{+r\kappa}r\) components are eliminated, the definite-parity requirement defines the energy levels \(E = -\kappa_n^2, \kappa_n = \frac{1}{2}(\lambda - 1 - n), n = 0, 1, \ldots, n_{\text{max}}, n_{\text{max}} = \text{integer part of } (\lambda - 1)\). By chance, the resulting bound-state solution

\[
\psi_n(r) = \text{const. ch}^{\kappa + 1 - \lambda} \frac{r}{2} {}_2F_1 \left( 2\lambda - n - 1, -n, \lambda - n, \frac{1 + \text{th}(r/2)}{2} \right)
\]

is polynomial — this is characteristic for the so called Bargman potentials \([6]\).

(B) The initial choice of the variable \(z = -\text{sh}^2(r/2)\) leads to the correct parity of solution from the very beginning. We must extend its convergence to the whole real axis \(r \in (-\infty, \infty)\) by analytic continuation. Finally, the asymptotic boundary conditions quantize the energy levels.

(C) In our approach, we satisfy first the asymptotic conditions at \(r \rightarrow \pm \infty\) by \(z = 1/\text{ch}^2(r/2) = 2(\alpha + \chi r), \alpha = 1\). The series (2.1) is convergent in the half-interval \(r \in (0, \infty)\) so that it is sufficient to match the two \(r > 0\) and \(r < 0\) branches together at \(r = 0\).

**Remark 1.** The values \(p = 0\) or \(p = 1\) correspond to the solutions of the even or odd parity, respectively.

*Proof* by analytic continuation is easy.

**Remark 2.** The matching (quantization) conditions read

\[
\lim_{r \rightarrow 0} \psi'(r) = 0 \quad \text{or} \quad \lim_{r \rightarrow 0} \psi(r) = 0
\]

for \(p = 0\) or \(1\), respectively, and are numerically well behaved.

*Proof. Let us fix some small \(r \neq 0\) and replace (2.3) by the equivalent relations

\[
{}_2F_1 \left( \kappa + \frac{1}{2}, \kappa + \frac{1 - \lambda}{2} + 1, 2 + 2\kappa, \frac{1}{\text{ch}^2(r/2)} \right) \left( \kappa + \frac{1 - \lambda}{2} \right) = 0,
\]

or

\[
(2.4)
\]