VARIATIONAL CHARACTERIZATION OF TORSIONLESS CONNECTIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We state a variational principle which allows the variational characterization of the class of torsionless affine connections on a Riemannian manifold, as well as of any subclass of it determined by a suitable set of constraints on the metricity of the connection.

1. INTRODUCTION

This paper is devoted to the study of gravitational variational principles in the presence of constraints. This is suggested by the remark that many variational problems do not yield all the necessary field equations. A well-known example is the Palatini principle [2], which requires some subsidiary conditions in order to determine the equations for both the metric and the connection, these quantities being regarded as independent variables.** In other words, recalling that a connection on a Riemannian manifold is determined by its torsion and metricity fields $T$ and $Q$, we may say that the variational principle in the arguments $\Phi$, $T$, $Q$ ($\Phi$ is the fundamental form) based on the Hilbert–Palatini Lagrangian alone does not provide a complete set of equations for $\Phi$, $T$ and $Q$.

This argument may be generalized to the study of variational principles, based on linear Lagrangians, in the presence of constraints. Among the various possible choices of the constraints, the most meaningful fall within two categories: algebraic constraints on the torsion tensor, and algebraic constraints on the metricity tensor. The first possibility, which admits the Palatini method as a special case, was examined by E. Massa [1].

The second possibility will form the subject of the present paper. We shall state a variational principle which always yields the condition $T = 0$ and the usual empty-space Einstein equations. No conditions on the metricity arise. In this sense, the variational principle characterizes the class of the torsionless connections over the manifold. Furthermore, the principle may be completed by whatever constraint on the metricity, thus allowing the variational characterization of arbitrary, a-priori chosen, subsets of the class of torsionless connections.

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**A discussion of some modifications of Palatini principle which do not present such a drawback may be found in [7–9], [1].

2. PRELIMINARIES

Let us consider a (pseudo) Riemannian four-dimensional manifold \( M \). Let \( \omega^j \) be a field of frames over \( M \), and \( \Phi = g_{ij} \omega^i \otimes \omega^j \) the fundamental form of \( M^* \). We recall that the knowledge of the coefficients of an affine connection \( \nabla \) is equivalent to the knowledge of two tensor fields, according to

\[
\Gamma^i_{jk} = \hat{\Gamma}^i_{jk} + Q^i_{jk} + W^i_{jk},
\]

where the \( \Gamma^i_{jk} \) are the coefficients of \( \nabla \), and the \( \hat{\Gamma}^i_{jk} \) are the coefficients of the Riemannian connection \( \hat{\nabla} \). The tensors \( Q \) and \( W \), called metricity and contortion fields, are defined by the equations

\[
Q^i_{jk} = \frac{1}{2} g^{im} (\nabla_k g_{mj} + \nabla_j g_{km} - \nabla_m g_{jk}) = Q^i_{jk},
\]

\[
W^i_{jk} = \frac{1}{2} (T^i_{jk} + T^i_{kj} + T^i_{jk}) = -W^i_{kj},
\]

\[
T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} \quad \text{(on a holonomic basis)}.
\]

Equations (2) may be inverted to give

\[
\nabla_k g_{jk} = Q^i_{jk} + Q_{kij},
\]

\[
T^i_{jk} = W^i_{jk} - W^i_{kj}.
\]

Equations (2–3) show that \( Q \) vanishes iff \( \nabla \) is metric, and that \( W \) vanishes iff \( T \) (the torsion field) vanishes itself. Moreover, according to Equations (2b, 3b), \( T \) and \( W \) may be indifferently used as independent variables.

In the following, the elementary theory of exterior algebra and analysis over \( M \) will be used [3]. In particular, we shall use the connection, contortion and metricity 1-forms

\[
\omega^i_k = \Gamma^i_{jk} \omega^j, \quad W^i_k = W^i_{jk} \omega^j, \quad Q^i_k = Q^i_{jk} \omega^j,
\]

the curvature two-forms

\[
\rho^i_k = d\omega^i_k + \omega^i_h \wedge \omega^j_k
\]

and the following forms, built up with the completely antisymmetric symbol \( \epsilon_{ijkh} \):

\[
\eta_{ij} = \frac{1}{2} \epsilon_{ijkh} |\det g_{ab}|^{1/2} \omega^k \wedge \omega^h, \quad \eta = \frac{1}{15} \eta_{ijk} \omega^j \wedge \omega^k.
\]

* Einstein's convention of summation over repeated indices is used throughout. Indices run from 1 to 4.

** In the following, the sign \( \sim \) will denote that the object over which it is superimposed is built up with the Riemannian connection.