Quantum Gravity and Schrödinger Equations on Orbifolds

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Abstract. The importance of conical singularities in the configuration space of gravitation theory is pointed out. Consideration is given to a possible inequivalence of intrinsic and extrinsic quantization and to the behaviour of the wave functional in the presence of such singularities.

It is well known that the set $M$ of configurations of a classical gravitational field is not a differentiable (infinite-dimensional) manifold. There are singular 'points' in $M$, corresponding to configurations which admit isometries and, hence, Killing vector fields. The objective of this short Letter is a physical discussion of the crucial role these singular points might play in a quantized version of gravitation theory. A more detailed exposition of these ideas is to be found in [1].

The set $\mathcal{M}(X)$ of all metric tensors $g_{\mu\nu}(x)$ on a manifold $X$ is acted upon by the group $\text{Diff} X$ of all diffeomorphisms of $X$ (coordinate transformations in a passive interpretation). Metric tensors which are coordinate transforms of each other correspond to the same (pseudo)Riemannian metric on $X$. The configuration space $M = \mathcal{M}(X)/\text{Diff} X$ obtained by dividing out the diffeomorphism group, does not have a smooth manifold structure. Rather, it is what is called an orbifold in the mathematical literature. Singularities may and will arise for metric tensors which admit isometries, or, in passive interpretation, are left fixed by some nontrivial coordinate transformation.

In the 3 + 1-dimensional formulation of gravitation theory, $M$ corresponds to the set of Riemannian metrics on three-dimensional manifolds. This is the situation we shall primarily have in mind. One could also take $X$ to be four-dimensional, and discuss the singularities in the set of pseudo-Riemannian structures on $X$, or the set of solutions of the Einstein equations on $X$. Finally, by replacing $\mathcal{M}(X)$ by the set of $G$ connections on $X$ and $\text{Diff} X$ by the group $\mathcal{G}$ of gauge transformations, applications to configurations of Yang–Mills fields are also possible [2].

The redundancy inherent in the description of gravitational fields by metric tensor fields also reflects itself in the canonical formulation. Nonpropagating degrees of freedom are eliminated by subsidiary conditions, which impose constraints both on the fields at fixed time, and on their time evolution. These constraints correspond to a vanishing condition for the momentum map associated with the diffeomorphism group. In addition, this formulation reveals the nature of the singularities located at symmetric fields: they turn
out to be conical just as, e.g., the zero set of a quadratic polynomial \( F(x) \) at \( x = 0 \), and they arise if one tries to solve for the constraint equations.

These canonical singularities originate in the phase space of the system, but they should generically project down to the configuration space \( \mathcal{M} \) as well.

One well-known consequence of the existence of such conical singularities is linearization instability [3]. Not every solution of the linearized constraint equations is tangent to a solution of the full problem. This makes linearization around symmetric solutions a rather dangerous procedure. In [4], it is shown how spurious solutions of the linearized equations can be eliminated by imposing additional quadratic constraints. In addition, a consistent quantization procedure for the linearized classical equation is given, in which the quadratic constraints are converted into operators. These operators have to vanish on physical Hilbert space states [5]. Fortunately, for technical reasons, this kind of linearization instability does not occur for linearizations about a Minkowski background. If it does occur, the quadratic subsidiary conditions imply that the linearization has to enjoy the same degree of symmetry as the field about which the linearization is performed. One proposed [5] explanation for this puzzling and unexpectedly high symmetry is provided by the Everett-Wheeler interpretation of quantum mechanics [6]. There, a higher degree of symmetry is obtained by incorporating the observer into the system. If one adopts the more conventional interpretation, the above-mentioned results mean that quantum fluctuations into less symmetric configurations are suppressed to linear order.

The quantization procedure for the linearized equations is an example of what might appropriately be called extrinsic quantization of a constrained classical theory: At first, all the classical quantities of the theory, including nonpropagating degrees of freedom, are converted into operators. Then, in a second step, constraints are implemented by transforming them into operators, where matrix elements between physical states have to vanish. The common way of treating constraints by manipulating a functional \( \delta \)-distribution in such a way as to convert it into an additional contribution to the action, also corresponds to an extrinsic approach. By writing

\[
\int D\phi \, \delta^{(\infty)}(f[\phi]) \, e^{iS[\phi]}
\]

one admits fluctuations about \( f[\phi] = 0 \), at least in intermediate steps.

In an intrinsic approach to quantization, one really has to eliminate unphysical degrees of freedom before quantizing. This is done by solving the constraint equations first and then quantizing only physical degrees of freedom. In terms of functional integrals, this means that the \( \delta^{(\infty)} \)-function really has to be integrated out.

This intrinsic kind of quantization, although conceptually attractive, is in general not performed because normally the constraint equations can be solved only locally. Even worse, the local solutions are nonpolynomial.