How Bad Can a Voting Location Be*

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Received May 22, 1985 / Accepted April 10, 1986

Abstract. Two questions concerning the location of a single facility by a voting process are investigated for transportation networks:

- What is the maximum number of customers located on the network preferring some rival point over a voting solution?
- How does the average distance of the customers to the facility increase when a voting location instead of a Weber location is chosen for the facility?

Complete answers are given for general networks and for certain planar networks, viz., the so-called generalized cacti.

1. Introduction

Suppose that voters located on a network must reach a decision upon locating a single facility. Each voter being a user of the facility wishes to have the facility as close as possible to his location. Hence the preference profile over the set of all potential locations is determined by the distances of the voters to the points of the network. Then, in a first attempt, majority voting may be applied in order to obtain a facility location that is unbeatable in an election. An outcome of the voting process is thus a Condorcet point, that is, a point \(c\) of the network such that no strict majority prefers another point to \(c\). The concept of a Condorcet solution though being very attractive has two serious drawbacks: first, Condorcet points need not exist on general networks; and second, even if Condorcet points exist they may be different from the Weber points (i.e., the absolute medians of the network). Location of the facility at a Weber point minimizes the average distance travelled by the users in

* This research was carried out during the First EURO Summer Institute on Location Theory held at Brussels, 1984. The paper benefitted from stimulating discussions with Pierre Hansen. The research of the second author was supported by the Action de Recherche Concertée of the Belgian Government under contract 84/8965.
order to reach the facility. If, however, the facility is located at a Condorcet point, then the average distance to the users may be almost three times larger than from a Weber location, see Hansen and Thisse (1981). These difficulties disappear when special networks are considered such as trees (i.e., networks without cycles). In fact, Hansen and Thisse (1981) have shown that every Condorcet point on a tree is a Weber point, and vice versa. A similar result, due to Wendell and McKelvey (1981), states that on a tree the Weber points coincide with the plurality points, i.e., points \( p \) such that the number of users preferring another point \( q \) to \( p \) is not larger than the number of users preferring \( p \) to \( q \). Cycles and, more generally, networks all of whose blocks are links or cycles (viz., cacti) may lack Condorcet points, but when a Condorcet point exists for a particular distribution of users, then Condorcet points and Weber points are necessarily the same, see Labbè (1983, 1985). Finally, one can precisely describe those networks for which Condorcet points always exist or coincide with the Weber points, respectively, see Bandelt (1985). In spatial location theory, by the way, positive results concerning the existence of Condorcet points are rare. One notable exception is the rectilinear plane: here every Weber point is also a Condorcet point (and vice versa), no matter how the users are located in the plane, see Wendell and Thorson (1974).

If in a particular situation a Condorcet point is not available, a natural way to tackle the problem is to look for a least objectionable solution. Any point of greatest voter agreement, i.e., a point \( s \) minimizing the maximum number of users preferring another point to \( s \), corresponds to this view. A solution concept of this kind was proposed in voting theory by Simpson (1969). We therefore refer to such minimax points \( s \) as Simpson points. The maximum relative rejection \( q(s) \), i.e., the maximum percentage of voters rejecting a Simpson location \( s \), means how objectionable this solution is. If \( q(s) \leq \frac{1}{2} \), then \( s \) is a Condorcet point. A value of \( q(s) \) very close to 1 indicates a high degree of circular voting.

Facility location resulting from a voting procedure has an immediate interpretation in competitive location theory, see Wendell and McKelvey (1981) and Hansen et al. (1986b). Consider two competing firms A and B which plan to establish a single shop each. A must establish its shop before firm B. Each customer will purchase one unit of commodity from the firm closest to his location; ties are broken in favour of firm A. Then a Simpson location will guarantee firm A a maximum number of customers regardless of where the competing firm B locates. Furthermore, if \( q(s) \leq \frac{1}{2} \) (respectively \( q(s) \geq \frac{1}{2} \)), then firm A (respectively firm B) has the advantage by entering first (respectively second) into the market. The smaller \( q(s) \), the larger the market share of firm A.

Notice that solution concepts related to the Simpson point have been considered by Slater (1975) and Hakimi (1983). Slater assumes that the indifferent customers are equally distributed between the two firms. He then defines a security center as a point \( p \) minimizing the maximum difference between the number of customers closer to another point \( q \) than to \( p \) and the number of customers closer to \( p \) than to \( q \). Hakimi (1983) considers the more general situation where firms A and B plan to establish \( k \) and \( m \) shops respectively. Then an optimal location for the shops of firm A is at a \((k|m)\)-centroid, that is, a set \( K \) of \( k \) points minimizing the maximum number of customers preferring a set \( M \) of \( m \) points to \( K \). Unfortunately, the problem of finding a \((k|m)\)-centroid is NP-hard when \( m > 1 \). In contrast, the set of all...