I. INTRODUCTION

In scanning the chapter on mathematical induction in most algebra texts one notices that almost all exercises begin with the words 'Show that ...' or 'Prove by mathematical induction that ...'. This formulation implies that the theorem to be proved is given by the author and the only thing the student has to do is to prove that the statement is true. Any brighter student will often ask himself, "How was the theorem discovered?", "What means lead to this conjecture and not to another one?", and "What might have been the steps which led the first discoverer to this theorem?". Very seldom do textbooks answer such questions.

In this article we shall introduce mathematical induction as it is used in the mathematical world. After an inductive investigation, a particular property, formula, or equation is formulated that appears to hold for each positive integer. Further analysis and calculations provide support for the validity of the conjecture. Now, and only now, do we employ the tool of mathematical induction to show that the result is valid for all positive integers. As a matter of fact, quite often the numerical computation clarifies to the student the true meaning of such an induction. That is, that the truth of any step implies the truth of the next one.

In this article we consider a variety of mathematical results. Most of the examples were tried out with considerable success by one or both of the authors in his teaching in secondary schools.

II. EMPIRICAL AND MATHEMATICAL INDUCTION

A dictionary explanation of a term is often a very useful way to clarify its meaning. Even though such explanations cannot be considered as mathematical definitions, because of their general circularity, nevertheless they can be very useful in the elucidation of some mathematical concepts.

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The concept of empirical induction as used in the natural sciences is explained by Webster's as "What has been observed in respect to a part, individual or species, may on the ground of analogy be affirmed or received of the whole to which it belongs".2

This induction approach to research is extensively used in experimental biology, chemistry, medicine, meteorology, and most other natural sciences, as well as in mathematics itself. The observation that a certain relationship between phenomena occurs a large number of times is considered a major justification for embedding this result in the theory of the given science.

In mathematics the empirical inductive approach may very often lead to major discoveries. To give one example, the German mathematician Gauss, the Prince of Mathematicians, by making an actual count, conjectured at the beginning of the 19th century, what is now known as the prime number theorem. This conjecture implies that the proportion of the primes among the integers from 1 to \( n \) is approximately equal to the ratio of 1 to the natural logarithm of \( n \). This conjecture was proved to be true only 90 years later, independently, by two mathematicians Hadamard and de la Vallée-Poussin, [1].3

However, in the realm of mathematics any statement which is only supported by an empirical inductive enquiry will have to be considered as a conjecture, an hypothesis which has a chance of being true. It remains a conjecture until a rigorous proof is found. One can list a number of statements supported by an empirical enquiry which eventually turned out to be false. Such examples often perplex and then later fascinate and intrigue students. Let us consider a few of these results.

One can easily prove, using simple algebra, that if \( 2^n - 1 \) is a prime then the exponent \( n \) must also be a prime. It is tempting to conjecture that the converse is also true: whenever \( n \) is a prime then \( 2^n - 1 \) also has to be a prime. Checking inductively this conjecture we find:

\[
2^2 - 1 = 3, \quad 2^3 - 1 = 7, \quad 2^5 - 1 = 31, \quad 2^7 - 1 = 127.
\]

All these cases support the conjecture.

However, \( 2^{11} - 1 = 2047 \) is not a prime as 2047 is divisible by 23.

Here we were lucky to have the fifth case contradict our conjecture.

Even more misleading is the following case. For ages mathematicians were searching for a formula which would help to generate prime numbers. About 500 B.C. Chinese mathematicians conjectured that if \( 2^n - 2 \) is divisible by \( n \)

3 These numbers in brackets refer to the references given at the end of the article.