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A Game-Based
Formal System for \( L_\infty \)

Abstract. A formal system for \( L_\infty \), based on a “game-theoretic” analysis of
the Łukasiewicz propositional connectives, is defined and proved to be complete.
An “Herbrand theorem” for the \( L_\infty \) predicate calculus (a variant of some work of
Mostowski) and some corollaries relating to its axiomatizability are proved. The
predicate calculus with equality is also considered.

§ 0. Introduction

This paper is a collection of results, related in that all are concerned
with the propositional and predicate calculus for the Łukasiewicz many-
valued logics.

In Giles ([2], [3]), a formal system for the propositional calculus is
implicitly defined. A set of axioms and rules are given, intended to cha-acterise the pairs of sequences \( A_1, \ldots, A_n | B_1, \ldots, B_m \) of formulas of
\( L_\infty \) such that \( \sum_{i=1}^{n} v(A_i) \geq \sum_{i=1}^{m} v(B_i) \) for all valuations \( v \) of the propositional
variables. In § 2 and § 3 below, we discuss the idea behind the system
and prove its completeness. This proof is based on an axiomatisation
of the valid formulas of \( L_\infty \) given by Łukasiewicz. The authors still have
no direct proof intrinsic to the system presented.

It is a simple consequence of the work in § 3 that it is decidable whe-
ther a given sequence \( A_1, \ldots, A_n | B_1, \ldots, B_m \) is valid, i.e. whether
\( \sum_{i=1}^{n} v(A_i) \geq \sum_{i=1}^{m} v(B_i) \) for all \( v \). In § 4 we prove this directly, applying some
arguments of MacNaughton [5]. This proof yields an explicit (though
none too good) time bound of \( 2^{cL^3} \), for some constant \( c \), with \( L \) the length
of a code for the position. Further, a very slight refinement of the argu-
ment yields a simple strengthening (Theorem 4.4) which is applied in § 5.

§ 5 is devoted to proof of some “Herbrand theorems” for the \( L_\infty \) pre-
dicate calculus. Crudely put, an “Herbrand theorem” for two-valued
logic associates effectively to each sentence \( \varphi \) a sequence \( \varphi_n, n \in \omega \) of quantifier-free sentences, such that \( \varphi \) is valid if and only if one of the \( \varphi_n \)'s
is valid. We cannot do so well for \( L_\infty \). What we can do is associate effectively
to each sentence \( \varphi \) a sequence \( \varphi_n, n \in \omega \), of quantifier-free sentences
such that \( \varphi \) is \( L_\infty \)-valid if and only if \( \varphi_n \)'s may be chosen “arbitrarily close
to \( L_\infty \)-valid.”

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The methods used to prove this are close to those used by Mostowski in [6] and [7] for a different purpose; where he used "epsilon functors", we use "Skolem functions". Further, we assume and use the compactness theorem he was attempting to prove in [7]. Because of this, the argument in § 5 is very close to the canonical semantic proof of the two-valued Herbrand theorem, and in fact is applicable uniformly to all the \( L_m \), \( m \in \omega \), as well.

We are also able to extend this "Herbrand theorem" to \( L_\infty \)-predicate calculi with equality, two notions of which we define. (Because of the lack of the Deduction Theorem in \( L_\infty \), the theorem for the predicate calculus with equality is not a corollary of the theorem for the predicate calculus without equality.) We are able in § 5 further to apply the Herbrand theorems to obtain upper bounds for the complexity of the set of valid sentences of the \( L_\infty \) predicate calculus with and without equality (and also for the \( L_m \) predicate calculi). For the predicate calculus without equality, these results appear in [6]. We feel that the argument here is worth presenting because of its transparency and because of the generalisation to languages with equality.

§ 1. Notation and prerequisites

Throughout we make free use of standard notations and notions of first-order logic. Any special notions required for many-valued logic are defined in the text. As usual, \( \omega = \{0, 1, 2, \ldots \} \) is the set of natural numbers, \( R \) the set of all real numbers, and \( I = [0, 1] \) is the closed interval of real numbers between 0 and 1.

This paper is not quite self-contained. In § 3 we use the completeness of Łukasiewicz's axiomatisation of \( L_\infty \), as proved by Wajsberg [11]. In § 4 we use some simple facts about finite polyhedra. In § 5, we apply the ultra-product construction to \( I \)-valued structures, as is done in Chang-Keisler [1].

§ 2. Let \( L \) be the propositional language built up out of a set \( P \) of propositional variables using the connectives \( F \) (which is 0-ary) \( \rightarrow, \lor, \land, \) and \( \neg \). A formula is prime if it is a \( P_n \) or \( F \). Other formulas are compound. A function from \( L \) to \( \omega \), 0 for all but finitely many arguments, is called a tenet. Let \( \mathcal{F} \) be the set of tenets; for \( \alpha, \beta \in \mathcal{F} \) let \( \alpha + \beta \) be the pointwise sum of \( \alpha \) and \( \beta \). Then \( \langle \mathcal{F}, + \rangle \) is the free abelian semigroup with \( L \) as generators. A prime tenet is a tenet nonzero only on propositional variables or \( F \). We write \( 0 \) for the zero of \( \mathcal{F} \).

An ordered pair \( \langle \alpha, \beta \rangle \) of tenets is called a position, henceforth to be written \( \alpha \upharpoonright \beta \). \( \alpha \upharpoonright \beta \) is a prime position if \( \alpha \) and \( \beta \) are prime tenets. If \( \Gamma_1 = \alpha_1 \upharpoonright \beta_1 \) and \( \Gamma_2 = \alpha_2 \upharpoonright \beta_2 \), we define \( \Gamma_1 + \Gamma_2 \) to be the position \( (\alpha_1 + \alpha_2) \upharpoonright (\beta_1 + \beta_2) \)