Transport in Macroscopically Inhomogeneous Materials

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We consider electrical and thermal transport in macroscopically inhomogeneous materials, when the two components forming a mixture have very different conductance properties. Because of their complexity, such systems are sometimes modeled by resistor networks. It is shown that the most natural models violate the Hashin-Shtrikman bounds to the effective conductivity of continuous composite materials. The distribution of the Joule heat between the phases, in the case of electrical conductance, is also largely erroneous. Thus, better estimates of conductance properties are obtained by disregarding detailed information about the phase geometry and instead using general methods for continuous materials, valid for a wide class of geometries.

KEY WORDS: composite materials; electrical conduction; resistor networks; thermal conduction.

1. INTRODUCTION

The effective conductivity of a macroscopically inhomogeneous two-phase material depends on the phase conductivities and on the concentration and geometrical distribution of the phases. However, it is usually difficult to evaluate the effective conductivity exactly, and approximate methods are required. Discrete models, in which few resistors in special configurations represent each grain, provide a drastic simplification.

The conductance properties of resistor networks, in which two resistors, \( R_1 \) and \( R_2 \), are distributed according to certain prescriptions, have been thoroughly studied from the point of view of critical behavior.

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and percolation properties when $R_1/R_2$ is zero or infinite \([1, 2]\). Yet little is known about the applicability of discrete resistor models to real composite materials. It is the purpose of this paper to discuss that question. In a previous paper \([3]\), dealing only with two dimensions, it was shown that natural discretizations violate certain absolute bounds to the overall conductivity of two-phase materials. The same behavior, illustrated by a specific example, is shown here to arise in three dimensions. Our presentation is in terms of the electrical conductivity, but it is relevant also for thermal conduction.

2. DISCRETIZATION MODELS

Consider a grain of a two-dimensional two-phase material. A reasonable description of the grain requires at least as many resistors as the number of neighboring grains (neglecting point-contacts). Figure 1 shows four such discretizations of a square grain. We call the resistor models in Figs. 1a, b, c, and d the cross, side, corner, and mid models, respectively. They are easily generalized to three dimensions. For instance, Fig. 2 shows the cross model of a cubic grain. The side model has 12 equal resistors, along the sides of a cube. The corner model has eight resistors, each one connecting a cube corner with the cube center. The mid model has 12 resistors, connecting the midpoints of adjacent cube sides. The number of resistors in each grain does not depend on the grain size.

Fig. 1. The cross (a), side (b), corner (c), and mid (d) resistor models of a square grain.