Scaling Transformations for \{0, 1\}-valued Sequences

G.L. O'Brien
Department of Mathematics, York University, Downsview Ontario, Canada M2N 3T5

Summary. Let \( r \geq 2 \) be an integer and let \( \varphi: \{0, 1\}^r \to \{0, 1\} \) be a function. Let \( T \) be the transformation on \( \Omega = \{0, 1\}^\mathbb{Z} \) given by \( (T\omega)(i) = \varphi(\omega(ri), \omega(ri + 1), \ldots, \omega(ri + r - 1)) \) for all \( i \in \mathbb{Z} \). For \( P \) in the class of strongly-mixing shift-invariant measures on \( \mathbb{Z} \), we investigate when \( P \) is invariant with respect to \( T \) and when \( T^nP \) converges. For example if \( r + 1 \) is odd and \( \varphi(\omega_0, \ldots, \omega_{r-1}) = 1 \) iff \( \sum \omega_i > \frac{1}{2}r \), the invariant measures are the Bernoulli measures with means 0, \( \frac{1}{2} \) or 1 and \( T^nP \) must converge to one of these three measures. Other choices of \( \varphi \) can give more complicated behaviour.

1. Introduction

Let \( \mathbb{Z} \) denote the set of integers and let \( \Omega = \{0, 1\}^\mathbb{Z} \). Let \( \mathcal{F} \) denote the \( \sigma \)-algebra of subsets of \( \Omega \) which is generated by the cylinder sets. Let \( S: \Omega \to \Omega \) be the shift transformation on \( \Omega \); that is \( (S\omega)(i) = \omega(i + 1) \). Throughout the following, measures on \( \Omega \) will be taken to be shift-invariant probability measures on the measurable space \((\Omega, \mathcal{F})\). We will consider \( \omega(i) \) to denote the \( i \)'th component of an element \( \omega \in \Omega \) or, when convenient, to denote the \( i \)'th term of a strictly stationary sequence \( \{\omega(i)\} \) of \( \{0, 1\} \)-valued random variables defined on some other probability space. In the latter case, one measure on \((\Omega, \mathcal{F})\) is the distribution of \( \{\omega(i)\} \). The reader should have no difficulty distinguishing between these two interpretations.

Let \( \varphi: \{0, 1\}^r \to \{0, 1\} \) be a function, where \( r \) is an integer greater than one. The scaling transformation corresponding to \( \varphi \) is the transformation \( T: \Omega \to \Omega \) defined by

\[
(T\omega)(i) = \varphi(\omega(ri), \omega(ri + 1), \ldots, \omega(ri + r - 1)).
\]

We call \( \varphi \) the scaling function for \( T \) and \( r \) the scale of \( T \). Let \( P \) be a measure on \( \Omega \) and let \( P_n \) be the measure on \( \Omega \) given by

\[
P_n(A) = T^nP(A) = P(T^{-n}(A)).
\]
The purpose of this paper is to determine which measures are invariant under $T$ (i.e., to solve the equation $P_i = P$) and to determine for which $P$ the sequence $\{P_n\}$ converges. The main results of the paper are given in Sect. 3, where we show that for $P$ in a large class, and for certain scaling transformations $T$, $P$ is $T$-invariant iff $P$ is one of a finite set of Bernoulli measures. The measures in the set are found by solving a polynomial equation obtained from $\varphi$. Furthermore, for such $T$, $P_n$ converges for all $P$ in the indicated class.

Of particular interest are the majority transformations obtained by taking $r$ odd and using the scaling function

$$\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1}) = \begin{cases} 1 & \text{if } \sum_{i=0}^{r-1} \omega_i > \frac{1}{2}r \\ 0 & \text{if } \sum_{i=0}^{r-1} \omega_i < \frac{1}{2}r \end{cases}$$

for all $(\omega_0, \ldots, \omega_{r-1}) \in \{0, 1\}^r$.

The question of identifying the invariant measures for majority transformations was posed by Professor F. Spitzer, who conjectured that the only non-trivial invariant measure is the Bernoulli measure with probability of each outcome being one half. In Sect. 4, we show that these transformations satisfy the requirements of our theorems and that Spitzer was essentially correct. In Sect. 6, we present several other invariant measures which are not in the large class to which our theorems apply.

Majority and other scaling transformations can be used to model a number of physical phenomena. Spitzer, motivated by the work of Mandelbrot (1977) on scaling, considered a person in an airplane observing the earth's surface below as being land or water. Incomplete resolution causes him only to detect the effect of successive intervals of length $r$. He records a one or zero (land or water) for each interval in a way that depends on the pattern of zeros and ones within the interval. The most natural situation would have him record the value that predominates (i.e. holds the majority) within the interval.

A similar example is obtained by considering a forester who walks through his domain and classifies each section as disease-ridden if the fraction of diseased trees he examines in the section exceeds some specified value.

A third example is obtained by considering a voting system in which the electorate is divided into cells of size $r$. Decisions are made by first voting within each cell and then having each cell (or its representative) vote in accordance with the majority within the cell. This could be extended to several stages by having cells, subcells, etc. (In practice, of course, representatives do not always heed the majority of their "cells".)

Finally, we note that Wilson (1979) has used a two-dimensional majority transformation to study the Ising model of magnetization.

Another interesting class of scaling transformations are the sum modulo two transformations obtained by defining

$$\varphi(\omega_0, \omega_1, \ldots, \omega_{r-1}) \equiv \sum_{i=0}^{r-1} \omega_i \pmod{2}$$

(4)