Forced vibrations and accumulation of plastic displacements of a compressed beam

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Summary: Using the theory of plastic hinges the equations of motion for a clamped beam made of elastic-plastic material are derived. As examples the response of the system to impact and harmonic excitation are discussed. The analytical results are compared with experimental investigations.

Wachstum plastischer Deformationen eines Druckstabes bei Zwangserregung


1 Coordinates and moment displacement relations

Consider a massless vertical clamped beam of length l and elastic-plastic properties, carrying a single mass m at its end. The objective is to set up an analytical method for predicting the response of the single mass system to a wide variety of forcing functions \( F(t) \). The instantaneous total displacement \( x(t) \) of the mass at time \( t \) is measured along the normal on the vertical (Fig. 1).

Assuming \( x \) to be small, all geometrical relations can be linearized.

\[
\begin{align*}
\text{Fig. 1. Coordinates}
\end{align*}
\]

The total displacement \( x(t) \) consists of two parts \( x_{\text{Rel}}(t) \) and \( x_{l}(t) \). The latter characterizes the inclination of the straight beam in a tensionless state. The known initial state is denoted \( x_{l}^{(0)} \) and can be looked upon as an imperfection. Movement of the system leads to an accumulation of plastic displacements and gives rise to a series of tensionless states \( x_{l} = x_{l}^{(0)}, x_{l}^{(1)}, \ldots, x_{l}^{(n)}, \ldots \) which depend on the nature of the applied forces \( F(t) \). The main objective of the paper is the determination of the growth of above tensionless states. Also measured is the relative coordinate \( x_{\text{Rel}} \) from an instantaneous state \( x_{f}^{(n)} \). Denoting \( w(s, t) \) as elastic deflection of the beam at the point \( s \) gives

\[
x_{\text{Rel}}(t) = w(l, t). \tag{1}
\]
Therefore
\[ x(t) = x_1(t) + x_{\text{Rel}}(t). \]  

The elastic analysis of a beam is concerned with deformations of the structure as a whole. In addition, it is necessary to pay attention to the local deformations if plastification occurs. In practice, plastification is spread over a finite portion of the beam. As an idealisation it is assumed that a plastic hinge is located at the fixed end of the beam. This simple assumption is in a good agreement with the experiments, described in the last chapter. To define the properties of such a hinge an arbitrary tensionless state \( x_j^{(n)} \) is considered. The bilinear dark curve in Fig. 2 shows qualitatively the bending moment \( M(0, t) \) in relation to the relative displacement \( x_{\text{rel}}(t) \) under the assumption of plastifications only in positive direction (accumulation). Fig. 2 gives the non-linear restoring moment in the sense of the vibration theory where \( x_{\text{rel}}(t) \) exists. Thus only two parameters describe the elastic plastic behavior of the beam i.e. the fully plastic moment \( M_p \) of the cross section at \( s = 0 \) and the limit elastic displacement \( x_e \) at \( s = l \).

![Fig. 2. Moment displacement curve during the existence of \( x_j^{(n)} \)](image)

It is further assumed that the elastic properties of a beam made of mild steel do not change during continuous load and unload cycles. Therefore \( M_p \) and \( x_e \) remain unchanged for any arbitrary \( x_j^{(n)} \). On the other hand, the statical equilibrium position \( x_{\text{st}}^{(n)} \) and the maximum amplitude \( A^{(n)} \), which both are measured relative to \( x_j^{(n)} \) vary during the sweep through the tensionless states \( x_j^{(0)}, x_j^{(1)} \ldots x_j^{(n)} \ldots \). The value \( x_{\text{st}}^{(n)} \) is achieved from static considerations, knowing the respective \( x_j^{(n)} \), while \( A^{(n)} \) can be obtained only by setting the dynamical criteria \( x_{\text{rel}} = 0 \). Plastification is terminated when the relative velocity of the plastic motion changes its sign. After this, the system moves again fully elastic but having a new tensionless state
\[ x_j^{(n+1)} = x_j^{(n)} + (A^{(n)} - x_e). \]  

The difference \( A^{(n)} - x_e \) is the increase in the plastic displacements from when \( x_j^{(n)} \) sweeps over to \( x_j^{(n+1)} \). This sequence for all \( n \) defines the accumulation function.

2 Equations of motion in the relative coordinates \( x_{\text{Rel}}^E \) and \( x_{\text{Rel}}^P \)

To set up the equations of motion during an instantaneous tensionless state \( x_j^{(n)} \) a period of elastic motion \(-x_e \leq x_{\text{Rel}} \leq x_e\) and a period of plastic motion \(|x_e| \leq |x_{\text{Rel}}| \leq A^{(n)} \) must be considered. For the elastic range the partial initial boundary value problem
\[
\begin{align*}
EIw'' + mgw' &= 0 \\
 w(0, t) &= w'(0, t) = w''(l, t) = 0 \\
 -x_e &\leq x_{\text{Rel}} \leq x_e
\end{align*}
\]

\[ \begin{cases} 
 w(0, t) = w'(0, t) = w''(l, t) = 0 \\
 EIw''''(l, t) = m\ddot{w}(l, t) - mg[x_j^{(n)}|l + w'(l, t)] - F(t) 
\end{cases} \quad -x_e \leq x_{\text{Rel}} \leq x_e \]  

is valid. A dash indicates derivatives with respect to the coordinate \( s \) along the beam. A dot is a time differentiation. Separating variables
\[ w = y(s) \varphi(t) \]  

(5)