The Local Growth of a Random Field

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Summary. We prove sufficient conditions such that almost every sample function \( X(t, \omega) \) of a random field satisfies the following infinite local growth condition:

\[
\limsup_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{\phi(|s-t|)} = \infty \quad \forall \, t, \quad \text{or}
\]

\[
\limsup_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{\phi(|s-t|)} = \infty \quad \forall \, t,
\]

where \( \phi \) is a modulus function. We then apply the general results to Gaussian random fields and obtain, among other results, a new result for the nowhere Hölder continuity of a Gaussian field with continuous paths.

§ 1. Introduction

In the study of the sample function behaviour of a random field \( X(t, \omega) \), we are interested in whether almost every \( X(\cdot, \omega) \) satisfies the following infinite growth condition:

\[
\limsup_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{\phi(|s-t|)} = \infty, \quad \forall \, t. \tag{1.1}
\]

Here, \( \phi(r) \) is a continuous increasing function which vanishes at \( r = 0 \) (e.g. \( \phi(r) = r^\gamma \), \( \gamma > 0 \)). Take \( \phi(r) = r^\gamma \); we see that (1.1) relates to the ever interesting problem of the nowhere Hölder continuity of a stochastic process with continuous paths.

It is known nowadays that the local time (or occupation density) is ideally suited to the study of (1.1). In fact, using the local time theory, we can discuss the stronger version of (1.1):

\[
\limsup_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{\phi(|s-t|)} = \infty, \quad \forall \, t. \tag{1.2}
\]
or even more strongly,
\[
\text{ap-lim sup}_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{\phi(|s - t|)} = \infty, \quad \forall t. \tag{1.3}
\]

In the above, "ap" stands for "approximate". Recall that (1.2) and (1.3) amount respectively to saying that, for each \(q > 0\), \(t\) is not a point of density and \(t\) is a point of dispersion for \(\{s: |X(s, \omega) - X(t, \omega)| \leq q \cdot \phi(|s - t|)\}\) (we refer to Geman-Horowitz [7, Appendix] for the definitions of ap-lim sup, ap-lim inf and ap-limit). Thus, (1.2) and (1.3) not only imply (1.1) but also provide us with the information on the "proportion" of \(s\)'s near \(t\) for which \(|X(s, \omega) - X(t, \omega)|/\phi(|s - t|)\) is large.

In this paper, we shall present sufficient conditions for which (1.2) and (1.3) hold (Theorem 1, §3). Our derivation is quite different from the standard approach appearing in Geman-Horowitz [7]; this is an excellent exposition on local time theory and its applications. Their method is first to take up the study of nonrandom fields, then the results are applied to random fields in a parallel development (see [7, §§10, 23 and 27]). However, our basic tool in the proof of the main results is a local time inequality (Proposition 1, §2) which appeared originally in Pitt [8]. Our results are based on bivariate density functions, and do not involve the concepts of the "AC-\(p\) condition" and the "local nondeterminism" (see [7, §§23, 27]; the original contributions are due respectively to Geman [6], Berman [3] and Pitt [8].) Because of the explicit expression of Gaussian bivariate density functions, we can easily apply our general results to Gaussian fields to obtain the following results (Theorem 2, 3, §3). Assume that \(X(t, \omega)\) is an \((N, d)\) Gaussian field with index \(\beta\) (see the definition in §3) and that \(N > \beta d\). If \(\gamma > \beta\), then almost every \(X(\cdot, \omega)\) satisfies
\[
\text{ap-lim sup}_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{|s - t|^{(N/d + \gamma)}} = \infty \forall t, \tag{1.4}
\]
\[
\text{ap-lim sup}_{s \to t} \frac{|X(s, \omega) - X(t, \omega)|}{|s - t|^{(N + \gamma)}} = \infty \forall t. \tag{1.5}
\]

The result (1.4) has a new implication for the nowhere Hölder continuity of Gaussian sample paths (Theorem 4, §3). We also indicate the interesting comparison of our results with the results obtained by using the "AC-\(p\) condition" and the "local nondeterminism" (Remark, §3).

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§ 2. An Inequality for the Local Time of a Random Field

In the following, \(X \equiv \{X(t, \omega)\}\) is always assumed to be a separable measurable \((N, d)\) random field. By an \((N, d)\) random field, we mean that \(X\) is a family of \(d\)-dimensional random vectors parametrized by \(t \in I = [0, 1]^N\). The local time (also