The Law of the Iterated Logarithm for Normalized Empirical Distribution Function

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This paper deals with the law of the iterated logarithm and its analogues for
\[ \sup_{x} |x - F_{n}(x)| (x(1-x))^{\frac{1}{2}}, \]
where the sup is taken on an interval of the form
\[ (a_{n}, b_{n}), \quad (0 < a_{n} < b_{n} < 1). \]
Under certain conditions on \( a_{n} \) and \( b_{n} \) the corresponding \( \lim \sup \) results will be proved.

1. Introduction

Let \( X_{1}, X_{2}, \ldots \) be a sequence of independent random variables each having the uniform distribution on \((0, 1)\) and denote by \( F_{n}(x) \) the empirical distribution function of the variables \( X_{1}, X_{2}, \ldots, X_{n} \).

In this paper we consider the process \( (F_{n}(x) - x) \tau(x) \) with some appropriate weight function \( \tau(x) \). Special attention will be paid to the normalizing weight function defined by

\[ \tau_{0}(x) = \{x(1-x)\}^{-\frac{1}{2}}. \tag{1.1} \]

Put

\[ \alpha_{n}(x) = (F_{n}(x) - x) \tau_{0}(x). \tag{1.2} \]

Then the process \( n^{\frac{1}{2}} \alpha_{n}(x) \) is normalized in the sense that for all integer \( n \geq 1 \) and all \( 0 < x < 1 \) we have

\[ E(\alpha_{n}(x)) = 0 \quad \text{and} \quad nE(\alpha_{n}^{2}(x)) = 1. \]


\[ n^{\frac{1}{2}} (2 \log \log n)^{-\frac{1}{2}} (F_{n}(x) - x) \tau(x) \]

provided that the weight function \( \tau(x) \) satisfies the condition

\[ \int_{0}^{1} \frac{\tau^{2}(x)}{\log \log \{x(1-x)\}^{-1}} \, dx < \infty. \tag{1.4} \]
In [10] it is shown that this condition is not only sufficient but also necessary in some sense. We state only the main corollary to this result as

**Theorem 1.1 (James).** If the weight function \( \tau(x) \) satisfies (1.4), then

\[
\limsup_{n \to \infty} \left( \frac{n}{\log \log n} \right)^{\frac{1}{3}} \sup_{0 < x < 1} |F_n(x) - x| \tau(x) = \sup_{0 < x < 1} \{2x(1-x)\tau(x)\}^{\frac{1}{2}} \quad \text{a.s.} \tag{1.5}
\]

**On the other hand, if the integral in (1.4) diverges, then**

\[
\limsup_{n \to \infty} \left( \frac{n}{\log \log n} \right)^{\frac{1}{3}} \sup_{0 < x < 1} |F_n(x) - x| \tau(x) = \infty \quad \text{a.s.} \tag{1.6}
\]

It is easy to see that for \( \tau_0(x) \) defined by (1.1), the integral in (1.4) diverges, hence (1.6) holds true for \( \tau_0(x) \). This statement follows also from a theorem of Baxter [1]. In fact we have shown [4]:

**Theorem 1.2.** Let \( a_n(x) \) be defined by (1.2) and put

\[
T_n = \sup_{0 < x < 1} |a_n(x)|. \tag{1.7}
\]

*If* \( \sum a_n = \infty \), then

\[
\limsup_{n \to \infty} (n a_n^2 T_n) = \infty \quad \text{a.s.} \tag{1.8}
\]

*If* \( \sum b_n < \infty \), then

\[
\limsup_{n \to \infty} (n b_n^2 T_n) = 0 \quad \text{a.s.} \tag{1.9}
\]

An interesting corollary to this result is

\[
\limsup_{n \to \infty} (n^{\frac{1}{3}} T_n) \left( \frac{\log \log n}{\log n} \right)^{-1} = e^{\frac{1}{2}} \quad \text{a.s.} \tag{1.10}
\]

Csörgő and Révész [6] raised the question, what can be stated about \( \sup |a_n(x)| \) where the sup is taken over \( a_n(x) \) with suitably chosen sequence \( \{a_n\} \). They proved

**Theorem 1.3 (Csörgő-Révész).** Let \( a_n = (\log n)^{\frac{1}{4}}/n \), and \( a_n(x) \) be defined by (1.2). Then

\[
\limsup_{n \to \infty} \left( \frac{n}{\log \log n} \right)^{\frac{1}{3}} \sup_{a_n < x < 1 - a_n} |a_n(x)| = 2 \quad \text{a.s.} \tag{1.11}
\]

Eicker [8] and Kiefer [11] study the behavior of \( F_n(x) \) along a sequence \( \{a_n\} \) and prove the following results:

**Theorem 1.4 (Eicker and Kiefer).** Assume that \( a_n < \frac{1}{2} \), and

\[
\lim_{n \to \infty} n a_n (\log n)^{-1} = \infty.
\]

Then

\[
\limsup_{n \to \infty} \left( \frac{n}{\log \log n} \right)^{\frac{1}{3}} |a_n(a_n)| = 2^{\frac{1}{6}} \quad \text{a.s.} \tag{1.12}
\]