Representation of an Isotropic Diffusion as a Skew Product*

By

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I. Introduction

The processes considered in this paper have state space $\mathbb{R}^3$ and are characterized by the following conditions:

(i) possess the simple Markov property
(ii) are homogeneous in time
(iii) have continuous paths
(iv) are isotropic
(v) do not pass through the origin at positive times except possibly on a set of paths of zero probability.

These conditions will be made precise in Sec. 2 after the basic notation has been presented.

The sample space of all continuous paths on $\mathbb{R}^3$ will be expressed as the Cartesian product $\Omega \times \Omega'$ of two sample spaces: $\Omega$ consists of all continuous paths $\omega$ on the radial coordinate space $[0, \infty)$, and $\Omega'$ of all continuous paths $\omega'$ on the spherical coordinate space $S^2$.

A diffusion of the type described is expressed by using spherical coordinates, as

$$x(t, \omega \times \omega') = [r(t, \omega), \varphi(t, \omega')]$$

where $r(t, \omega), \omega \in \Omega$, is the radial motion and $\varphi(t, \omega'), \omega' \in \Omega'$ is the spherical motion both associated to $x(t, \omega \times \omega')$.

It is aimed to prove these results:

a) The radial process $r(t, \omega)$ is simple Markov and homogeneous in time.

b) $x(t, \omega \times \omega')$ can be represented as the so-called skew product of the radial process and an independent spherical Brownian motion $\mathfrak{B}$ run with a clock $\sigma(t, \omega)$ depending on the radial path $\omega$. That is, it will be shown that, with probability one for all $t$ simultaneously:

$$x(t, \omega \times \omega') = (r(t, \omega); \mathfrak{B}[\sigma(t, \omega), \omega']).$$

c) $\sigma(t, \omega)$ is a non-negative, continuous non-decreasing function of $t$ for each fixed $\omega$. For each fixed $t$ it is measurable with respect to the sub-$\sigma$-field determined

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by the radial motion up to time $t$. Moreover it accomplishes the following additive property:

$$\sigma(t, \omega) = \sigma(t - s, \omega^+_{s}) + \sigma(s, \omega) \text{ for } s < t$$

with probability one for all pairs $(s, t)$ simultaneously.

$\omega^+_{s}$ in (1.3) is the path defined by the equation:

$$r(t, \omega^+_{s}) = r(t + s, \omega).$$

In Sec. 3, by considering Green operators, the problem is reduced to the computation of the characteristic functional of $\varphi(t, \omega')$ as indicated in (3.21). Sec. 4 introduces a special Markov property for $\varphi(t, \omega')$. In Sec. 5 the characteristic functional of $\varphi(t, \omega')$ is actually computed and the desired expression (3.21) is obtained except for a term that still must be proved to be zero. In this proof, some ideas from [8] and [3] are used, although special arguments have to be applied due to the fact that $S^2$ is not a group and also in order to show that the clock $\sigma(t, \omega)$ is finite. Sec. 6 leads to the construction of a spherical process with the characteristic functional of $\varphi(t, \omega')$ and makes the above-mentioned term correspond to interlarded Poisson jumps in a spherical Brownian motion run with a suitable clock. Sec. 7 proves the equivalence of $\varphi(t, \omega')$ with the process constructed in Sec. 6 by applying the special Markov property of Sec. 4, and produces the final result.

2. Notation and basic definitions

The sample spaces $\Omega$ and $\Omega'$ already have been introduced in Sec. 1. $\Omega$ consists of all continuous functions $\omega$ from $[0, \infty)$ into $[0, \infty)$, and $\Omega'$ of all continuous functions $\omega'$ from $[0, \infty)$ into $S^2$.

Throughout this paper $\mathcal{B}$ is the Borel $\sigma$-field of subsets of $\Omega \times \Omega'$ generated by sets of the type:

$$(\omega \times \omega': x(t, \omega \times \omega') \in A), \ A \text{ Borel set } \subset \mathbb{R}^3.$$  

$\mathcal{B}_t$ is the sub-$\sigma$-field of $\mathcal{B}$ whose generators are those in the definition of $\mathcal{B}$ in which $t \leq s$. Analogously $\mathcal{B}_{(s_1, s_2)}$ is that sub-$\sigma$-field whose generators have $s_1 < t \leq s_2$.

$\mathcal{B}_r$ is the sub-$\sigma$-field generated by sets of the type:

$$(\omega \times \omega': r(t, \omega) \in A_1), \ A_1 \text{ Borel set } \subset [0, \infty).$$  

$\mathcal{B}_\varphi$ is the sub-$\sigma$-field generated by sets of the type:

$$(\omega \times \omega': \varphi(t, \omega') \in A_2), \ A_2 \text{ Borel set } \subset S^2,$$

where $S^2$ is the unit sphere in $\mathbb{R}^3$.

For each point $a \in \mathbb{R}^3$ the process defines a probability measure $P_a(B)$ for all $B \in \mathcal{B}$. This means the probability that a continuous path starting at $a$ belongs to the Borel set $B$. A dot will often be used for a generic point in $\mathbb{R}^3$, for instance, $P_a(B)$. It is understood that, whenever "." is used several times in an argument, it always refers to the same point.