Incompressible Reiner-Rivlin Fluids Obeys the Orthogonality Condition*

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Summary: The orthogonality principle recently proved by one of the authors is applied to the constitutive equation of the incompressible Reiner-Rivlin fluid. By means of a few examples it is shown that the corresponding simpler fluid still exhibits the typical cross effects of the Reiner-Rivlin fluid.


1. Introduction

In a recent article Ziegler [1] has proved an orthogonality principle in Thermodynamics of irreversible processes, which, incidentally, is equivalent to various extremum theorems [2—7]. Since most material flows are irreversible, the principle may be expected to affect many problems arising in Continuum Mechanics. Conversely, one may hope to test the principle by comparing the predictions in this field with the results of experiments.

This paper is a first step in this direction. Applying the orthogonality condition to an incompressible Reiner-Rivlin fluid [8, 9], the authors confirm a former result [10] restricting the usual form of the constitutive equation. Developing the stresses obtained in this manner in terms of powers of the deformation rates, they show that the first two approximations yield the Newtonian fluid, that the third one defines a quasilinear fluid and that the fourth approximation yields a truly nonlinear fluid exhibiting the characteristic cross effects observed in certain experiments. A few typical examples confirm that the effects predicted in these cases by the theory of Reiner-Rivlin fluids may be fully explained by the simpler theory based on the orthogonality principle.

To avoid misunderstandings: the authors are aware of the existence of more general fluids than those of the Reiner-Rivlin type, and of the desirability to study the implications of the orthogonality principle for these more general cases. They believe, however, that it is reasonable to start with a simple situation.

2. The Constitutive Equation

In an incompressible fluid the deformation rate, defined as the symmetric part of the velocity gradient, has the basic invariants

\[ d_{ij} = \frac{1}{2} \left( \nu_{ik} + \nu_{ki} \right) \tag{2.1} \]

of the velocity gradient, has the basic invariants

\[ d_{(1)} = d_{ii} = 0, \quad d_{(2)} = \frac{1}{2} d_{i i} d_{ij}, \quad d_{(3)} = \frac{1}{3} d_{i j} d_{k i} d_{k j}. \tag{2.2} \]

The stresses \( \sigma_{ij} \) may be split up, according to [1], into a reversible stress

\[ \sigma_{ij}^{(r)} = - \bar{p} \delta_{ij}, \tag{2.3} \]

where \( \bar{p} \) is the indeterminate hydrostatic pressure, and an irreversible stress which [see, e. g., [11] p. 131], in the case of a Reiner-Rivlin fluid, has the form

\[ \sigma_{ij}^{(i)} = g \left( d_{(2)}, d_{(3)} \right) d_{ij} + h \left( d_{(2)}, d_{(3)} \right) \left( d_{ik} d_{kj} - \frac{2}{3} d_{ik} d_{kj} \right), \tag{2.4} \]

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where the functions \(g\) and \(h\) are only subject to the condition that the rate of dissipation work,
\[
L = \frac{1}{\rho} \frac{\partial}{\partial t} \sigma^{(i)}_{ij} d_{ij},
\]
referred to the unit mass by means of the density \(\rho\), is nonnegative. Expressing (2.5) in terms of the deformation rate, one obtains the dissipation function \(D(d_{ij})\), which is positive definite and, on account of (2.5), (2.4) and (2.2), has the form
\[
D = \frac{1}{\rho} (2 g d_{(2)} + 3 h d_{(3)}).
\]

The orthogonality principle [1] permits us to express the two functions \(g(d_{(2)}, d_{(3)})\) and \(h(d_{(2)}, d_{(3)})\) by a single one, \(D(d_{(2)}, d_{(3)})\). Since the \(d_{ij}\) are the “velocities” and the \(\sigma_{ij}^{(t)}\) the corresponding “irreversible forces” per unit mass, the principle requires that
\[
\sigma_{ij}^{(t)} = q \left( \frac{\partial D}{\partial d_{ij}} \right) d_{kl}.
\]
From (2.2) we obtain
\[
\frac{\partial d_{(2)}}{\partial d_{ij}} = d_{ij}, \quad \frac{\partial d_{(3)}}{\partial d_{ij}} = d_{kl} d_{kj} - \frac{1}{3} d_{(2)} d_{ij},
\]
where the last term is due to the fact that \(d_{ij}\) (see [10], p. 382) is a deviator, subject to the condition \(d_{ii} = 0\). It follows that
\[
\frac{\partial D}{\partial d_{ij}} = \frac{\partial D}{\partial d_{(2)}} d_{ij} + \frac{\partial D}{\partial d_{(3)}} \left( d_{kl} d_{kj} - \frac{2}{3} d_{(2)} d_{ij} \right).
\]

Substitution in (2.7) yields the relation
\[
\sigma_{ij}^{(t)} = q D \left( 2 \frac{\partial D}{\partial d_{(2)}} d_{(2)} + 3 \frac{\partial D}{\partial d_{(3)}} d_{(3)} \right) \frac{1}{2} \left[ \frac{\partial D}{\partial d_{(2)}} d_{(2)} + \frac{\partial D}{\partial d_{(3)}} \left( d_{kl} d_{kj} - \frac{2}{3} d_{(2)} d_{ij} \right) \right]
\]
already obtained in [10].

Comparing (2.10) with (2.4) we confirm that the orthogonality principle restricts the field of Reiner-Rivlin fluids. In fact, the two constitutive equations become identical if we set
\[
g = q D \left( 2 \frac{\partial D}{\partial d_{(2)}} d_{(2)} + 3 \frac{\partial D}{\partial d_{(3)}} d_{(3)} \right) \frac{1}{2} \left[ \frac{\partial D}{\partial d_{(2)}} d_{(2)} + \frac{\partial D}{\partial d_{(3)}} \left( d_{kl} d_{kj} - \frac{2}{3} d_{(2)} d_{ij} \right) \right],
\]
reducing hereby the two arbitrary functions to a single one.

### 3. Expansion

For a more detailed discussion of the orthogonality principle’s implications on constitutive equations of the form (2.4) we limit ourselves to dissipation functions which, for small values of \(d_{ij}\), may be expanded in power series of their arguments. In view of (2.6) we start with the expansions
\[
\begin{align*}
& g(d_{(2)}, d_{(3)}) = g_0 + g_2 d_{(2)} + g_4 d_{(2)}^2 + g_6 d_{(2)}^3 + \cdots, \\
& h(d_{(2)}, d_{(3)}) = h_0 + h_2 d_{(2)} + h_4 d_{(2)}^2 + \cdots,
\end{align*}
\]
which yield
\[
q D = 2 g_0 d_{(2)} + 3 h_0 d_{(3)} + 2 g_2 d_{(2)}^2 + (2 g_3 + 3 h_2) d_{(2)} d_{(3)} + 2 g_4 d_{(2)}^3 + 3 h_3 d_{(2)}^3 + \cdots.
\]
According to (2.2) \(d_{(2)}\) and \(d_{(3)}\) are small of the second and third order, respectively; the terms written out on the right-hand side of (3.2) thus range from orders 2 to 6 in \(d_{ij}\).

Differentiation of (3.2) yields
\[
q \frac{\partial D}{\partial d_{(2)}} = 2 g_0 + 4 g_2 d_{(2)} + (2 g_3 + 3 h_2) d_{(2)} d_{(3)} + 6 g_4 d_{(2)}^2 + \cdots, \quad q \frac{\partial D}{\partial d_{(3)}} = 3 h_0 + (2 g_3 + 3 h_2) d_{(2)} + 6 h_3 d_{(2)}^2 + \cdots.
\]