Relative Equilibrium Positions and Their Stability
for a Multi-Body Satellite in a Circular Orbit

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Summary: An arbitrary number of gyrostats is interconnected by ball-and-socket joints in a tree-configuration. The system is moving as a satellite in a circular orbit. Relative equilibrium positions and criteria for their stability are obtained from the first and second variations of the dynamic potential energy.


1. Introduction

The problem of relative equilibrium in a circular orbit in an inverse-square gravitational field is investigated. The satellite under consideration is a system of an arbitrary number $n$ of gyrostats which are interconnected in a tree-configuration by $n - 1$ frictionless ball-and-socket joints. Each gyrostat — in the following called body — consists of a rigid carrier and of free and controlled rotors whose axes are fixed in the carrier in arbitrary directions. To a free rotor no driving, friction or other torques about the axis are transmitted from the carrier. A controlled rotor has a constant angular momentum relative to the carrier. The initial conditions of the free rotors and the relative angular momenta of the controlled rotors are parameters which influence the relative equilibrium positions of the satellite and their stability. On body number $i$ ($i = 1, ..., n$) there are $v_i$ free rotors and one controlled rotor (whose relative angular momentum can be thought of as the resultant relative angular momentum of an arbitrary number of controlled rotors). It is assumed that the coupling between librational motions of the satellite and the motion of its composite center of mass is negligible and that the center of mass is moving in a circular orbit of radius $r$ with the orbital angular velocity $\omega_0$. The absolute value of $\omega_0$ is related to $r$ by the equation

$$\omega_0 = \sqrt{\frac{\mu}{r^3}}$$

(1)

with $\mu$ being the product of the universal gravitational constant and the mass of the attracting celestial body. Fig. 1 shows schematically a satellite in its circular orbit and the orbital reference frame.
frame with unit base vectors $e_1, e_2, e_3$ relative to which librational motions and equilibrium positions will be considered. The base vectors $e_2$ and $e_3$ are directed parallel to $\omega_0$ and along the local outward vertical, respectively.

Relative equilibrium positions and criteria for their stability will be developed as follows (s. [1, 2, 3] for the theoretical background). The generalized coordinates of the system are split in three groups. The first group is describing the attitude of the carriers relative to the orbital frame of reference. The second group is describing the angular positions of the free rotors relative to the carriers and the third group the angular positions of the controlled rotors relative to the carriers. Of the second and third group not the generalized coordinates themselves but only the generalized velocities enter the Lagrangian of the system. Those of the third group are constant parameters by assumption. Those of the second group will be eliminated by constructing the Routhian function. The Routhian contains as new constant parameters the generalized momenta which correspond to the cyclic coordinates. It represents the Lagrangian of a reduced system which has as generalized coordinates only the attitude coordinates of the first group. In the case of a circular orbit centrifugal forces have a potential. The reduced system is, therefore, conservative. Its potential energy $W$ is called dynamic potential energy of the original system. Relative equilibrium positions are found from the condition that the first variation of $W$ vanishes, and for stability it is sufficient that the second variation of $W$ is positive definite with respect to variations of the attitude coordinates.

2. The Kinetic Energy

In this section an expression is developed for the kinetic energy of the system which is needed for the formulation of the Lagrangian. Fig. 2 shows the system of bodies with the radius vector $r$ pointing from the center of gravity which is considered to be fixed in inertial space to the composite center of mass. From there the radius vector $r_k$ leads to the center of mass of body number $k$ ($k = 1, \ldots, n$). This center of mass is fixed in the carrier number $k$ since the mass distribution of the body is not affected by the rotation of the rotors. Finally, $\varrho$ is the vector from the body center of mass to a mass particle $dm$ on the same body. If the mass particle belongs to a rotor one must distinguish between its velocity $v$ relative to the carrier and the absolute velocity $\dot{r} + r + \omega + v$ of the coinciding point of the carrier. In these terms the kinetic energy $T$ is determined by the equation

$$2T = \sum_{k=1}^{n} m_k \int (\dot{r}_k + r_k + \omega + v)^2 \, dm .$$

The mass $m_k$ is including the masses of the carrier number $k$ and of all $v_k + 1$ rotors mounted on it. Multiplying out and recognizing the equations $\int \varrho \, dm = 0$, $\sum_{k=1}^{n} r_k m_k = 0$ and $\int v \, dm = 0$