Locally Normal Symmetric States
and an Analogue of de Finetti’s Theorem*

R.L. Hudson and G. R. Moody

Mathematics Department, University of Nottingham, University Park,
Nottingham NG7 2RD, England

§ 1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{H}_n$ be the Hilbert space tensor product of $n$ copies of $\mathcal{H}$. Let $\mathfrak{B}, \mathfrak{B}_n$ be respectively the $C^*$-algebras of all bounded operators on $\mathcal{H}, \mathcal{H}_n$. The infinite $C^*$-tensor product of countably many copies of $\mathfrak{B}$, denoted by $\mathfrak{B}^\otimes$, is defined as the $C^*$-inductive limit of the sequence $\{\mathfrak{B}_n\}$, equipped with the injections

$$\varphi_{nm}: A_n \mapsto A_n \otimes 1_{m-n}$$

from $\mathfrak{B}_n$ into $\mathfrak{B}_m$, where for $m > n$, $1_{m-n}$ is the identity operator on the Hilbert space tensor product of $m-n$ copies of $\mathcal{H}$. A state $\omega$ of $\mathfrak{B}^\otimes$ determines a hierarchy $\{\omega_n\}$ of states of the $C^*$-algebras $\mathfrak{B}_n$ by

$$\omega_n = \omega \circ i_n$$

(1)

where $i_n$ is the canonical injection of $\mathfrak{B}_n$ into $\mathfrak{B}^\otimes$. This hierarchy is consistent in the sense that, for $m > n$,

$$\omega_n = \omega_m \circ \varphi_{nm}.$$  

(2)

Conversely every consistent hierarchy $\{\omega_n\}$ of states $\omega_n$ of $\mathfrak{B}_n$ determines a state $\omega$ of $\mathfrak{B}^\otimes$. The state $\omega$ of $\mathfrak{B}^\otimes$ is called symmetric if each of the corresponding states $\omega_n$ is symmetric in the sense that

$$\omega_n(A_1 \otimes \cdots \otimes A_n) = \omega_n(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(n)})$$

for each choice of $A_1, \ldots, A_n \in \mathfrak{B}$ and each $\pi \in S_n$, the permutation group of $\{1, \ldots, n\}$, and locally normal if each state $\omega_n$ is normal, meaning that there exists a density

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operator, i.e. a positive operator $\rho_n$ of unit trace, such that
\[
\omega_n(A) = \text{tr} A \rho_n
\]
for all $A \in \mathcal{B}_n$.

Størmer [1] has investigated symmetric states of infinite tensor products of copies of an arbitrary $C^*$-algebra, showing that such states comprise a Choquet simplex, the extreme points of which are the product states, for which the $\omega_n$ have the property
\[
\omega_n(A_1 \otimes \cdots \otimes A_n) = \omega_1(A_1) \cdots \omega_1(A_n).
\]
This permits an arbitrary symmetric state to be represented as an integral over such product states, and provides a certain non-commutative analogue of de Finetti's theorem [2], as generalised by Hewitt and Ross [3].

In the present paper we shall modify Størmer's result to the case of locally normal symmetric states on $\bigotimes \mathcal{B}$. By Gleason's theorem [4], normal states are co-extensive with probability measures on the lattice of closed subspaces of the underlying Hilbert space, and our result is thus a natural generalisation of de Finetti's theorem from the viewpoint of the lattice theoretic approach to non-commutative probability theory.

In addition we investigate so-called *Bose-Einstein* locally normal symmetric states, which are characterised by the property that the density operators $\rho_n$ representing the states $\omega_n$ according to (3) satisfy the symmetry condition
\[
U_n \rho_n = \rho_n
\]
where for each $\pi \in \mathcal{S}_n$, $U_\pi$ is the unitary operator in $\mathcal{H}_n$ whose action on product vectors is
\[
U_\pi(x_1 \otimes \cdots \otimes x_n) = x_{\pi_1} \otimes \cdots \otimes x_{\pi_n}.
\]
It is shown that for such states the integral representation is carried by product states determined by normal states of $\mathcal{B}_1$ which are extreme in the convex set of all normal states of $\mathcal{B}_1$ (that is by pure normal states of $\mathcal{B}_1$).

In §2 we re-derive Størmer's characterisation for the case in question of the extreme point set of the set of all symmetric states as the product states, with the consequent integral representation over product states. The result is not new but the method of proof, related to the argument of Hewitt and Savage [3] for the commutative case, may be of some interest. In §3 we consider locally normal symmetric states, establishing a precise analogue of de Finetti's theorem from the lattice-theoretic probability viewpoint. In §4 the case of locally normal symmetric states with Bose-Einstein symmetry is considered.

**§ 2. Integral Representation of Symmetric States**

We denote by $S$ the set of all symmetric states of $\bigotimes \mathcal{B}$. Thus a state $\omega$ belongs to $S$ if and only if for all $n = 1, 2, \ldots$, $\pi \in \mathcal{S}_n$ and $A_1, \ldots, A_n \in \mathcal{B}$
\[
\omega(\iota_n(A_1 \otimes \cdots \otimes A_n)) = \omega(\iota_n(A_{\pi_1} \otimes \cdots \otimes A_{\pi_n})).
\]