Delphic Semi-Groups, Infinitely Divisible Regenerative Phenomena, and the Arithmetic of p-Functions

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1. Delphic Semi-groups

1.1 Introduction

In the first part of this paper we discuss what we shall call delphic semi-groups; these are (very roughly) commutative topological semi-groups which obey the central limit theorem for triangular arrays, and we shall prove that such a semi-group necessarily has an arithmetic very similar to the convolution arithmetic of distribution functions on \( \mathbb{R} \).

The present investigation arose from a study of the arithmetic of the multiplicative semi-group of renewal sequences, an account of which will be found in the published version [9] of my contributions to the Loutraki Symposium on Probability Methods in Analysis. A brief sketch of the general theory of delphic semi-groups was also given at Loutraki, but is not included in [9], and reference to that paper will be unnecessary unless the reader wishes to see the ultimate details of a simple concrete example. Another (but more sophisticated) example will be presented here in \( \S \) 2. The principal results of \( \S \) 1 of the present paper were announced without proofs in [10].

1.2 Definitions

Let \( \mathcal{G} \) be a commutative semi-group with a (unique) neutral element \( e \), and let \( \mathcal{G} \) carry a topology such that the mapping \((u, v) \rightarrow uv\) is continuous and \( \mathcal{G} \) itself is Hausdorff. Suppose further that \( \mathcal{G} \) is provided with a continuous homomorphism \( \Delta \) to the additive semi-group of non-negative real numbers, so that

\[
\Delta(uv) = \Delta(u) + \Delta(v) \quad (1)
\]

and

\[
0 \leq \Delta(u) < \infty. \quad (2)
\]

Obviously \( \Delta(e) \) must be zero, and while in general \( \Delta \) might vanish on other elements of \( \mathcal{G} \), we shall exclude this possibility by making the assumption

\( \Delta(u) = 0 \) if and only if \( u \) is the neutral element \( e \).

It is to be noted that \( \Delta \) need not be an isomorphism. The topological and algebraic structures on \( \mathcal{G} \) will be further related by the assumption

\[
\Delta(u) = 0 \quad \text{if and only if} \quad u = e.
\]
(B) for each $u$ in $\mathcal{G}$, the factors of $u$ form a compact set.

By a triangular array we shall understand a system

$$
\begin{align*}
&u(1, 1), \\
&u(2, 1), \quad u(2, 2), \\
&u(3, 1), \quad u(3, 2), \quad u(3, 3), \\
&\cdots \quad \cdots \quad \cdots \quad \cdots
\end{align*}
$$

of elements of $\mathcal{G}$, and we shall call

$$
u(i, 1)u(i, 2)\ldots u(i, i)
$$

the $i$-th marginal product for the array. We shall call an array

(1°) convergent when the marginal products converge to some $u$ in $\mathcal{G}$;

(2°) null when $A(u(i, j))$ converges to zero as $i$ tends to infinity, uniformly for $1 \leq j \leq i$.

We shall say that $u$ in $\mathcal{G}$ is indecomposable when it has both the properties

(i) $u \neq e$,

and

(ii) $u = vw$ implies that one of $v$ and $w$ is $e$ (the other factor therefore being equal to $u$),

and we shall say that $u$ is infinitely divisible when it possesses a $k$-th root in $\mathcal{G}$, for every $k = 2, 3, \ldots$. The neutral element is necessarily infinitely divisible (and because of the assumptions about $\Delta$ it has no factor distinct from itself); there may be no indecomposable elements. If $u$ in $\mathcal{G}$ is not indecomposable then we shall call it decomposable; thus $e$ has this property, and so has every infinitely divisible $u$.

Finally we make the assumption

(C) if a null triangular array is convergent, then the limit is infinitely divisible.

If $\mathcal{G}$ can be associated with a homomorphism $\Delta$ in such a way that the pair $(\mathcal{G}, \Delta)$ has all these properties, then $\mathcal{G}$ will be called a delphic semi-group. Assumption (C) makes precise the loose statement in §1.1 that our semi-groups will be required to obey the central limit theorem. We shall now show that such semi-groups possess all the general properties discovered by Khintchine in his study [11] of the arithmetic of distribution functions on $\mathbb{R}$.

In order to show that delphic semi-groups do exist we mention the

Example. Take $\mathcal{G}$ to be the half-open interval $(0, 1]$ with multiplication as the binary operation and the usual topology. Take $A(u) = -\log u$. Then (A) holds trivially, and $\{u' : u' \parallel u\}$ is the compact interval $[u, 1]$, so that (B) holds. Finally, every element is infinitely divisible (and so none is indecomposable), so that (C) also holds trivially. Thus our axioms are consistent.

1.3 The Analogues of Khintchine’s Theorems

First we have the trivial

**Theorem I.** An infinitely divisible element $u$ of $\mathcal{G}$ can always be represented as the limit of a convergent null triangular array.