Non-Central Limit Theorems for Non-Linear Functionals of Gaussian Fields *

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Dedicated to Professor Leopold Schmetterer on his sixtieth Birthday

Summary. Let a stationary Gaussian sequence $X_n$, $n = \ldots, -1, 0, 1, \ldots$ and a real function $H(x)$ be given. We define the sequences $y_n^N = \frac{1}{A_N} \sum_{j=(n-1)N}^{nN-1} H(X_j), n = \ldots, -1, 0, 1, \ldots; N = 1, 2, \ldots$ where $A_N$ are appropriate norming constants. We are interested in the limit behaviour as $N \to \infty$. The case when the correlation function $r(n) = E X_0 X_n$ tends slowly to 0 is investigated. In this situation the norming constants $A_N$ tend to infinity more rapidly than the usual norming sequence $A_N = \sqrt{N}$. Also the limit may be a non-Gaussian process. The results are generalized to the case when the parameter-set is multi-dimensional.

1. Introduction

Let a stationary Gaussian sequence $X_n$, $n = \ldots, -1, 0, 1, \ldots$ and $E X_n = 0, E X_n^2 = 1$ be given. We assume that the correlation function $r(n) = E X_0 X_n$ satisfies the relation

$$r(n) = n^{-\alpha} L(n), \quad 0 < \alpha < 1,$$

where $L(t), t \in (0, \infty)$ is a slowly varying function; i.e.

$$\lim_{s \to \infty} \frac{L(st)}{L(s)} = 1 \quad \text{for every } t \in (0, \infty),$$

and $L(t)$ is integrable on every finite interval. (See e.g. [5] Appendix 1.) We consider a real function $H(x)$ such that $H(x)$ does not vanish on a set of positive measure,

* This paper contains results closely connected to those of the paper by Taqqu, Z. Wahrscheinlichkeitstheorie verw. Gebiete 50, 53-83 (1979). The investigations were done independently and at about the same time. Different methods were used.
\[ \int_{-\infty}^{\infty} H(x) \exp \left( -\frac{x^2}{2} \right) dx = 0, \]

(1.3)

and

\[ \int_{-\infty}^{\infty} [H(x)]^2 \exp \left( -\frac{x^2}{2} \right) dx < \infty. \]

(1.4)

Throughout this paper \( H_j(x) \) denotes the \( j \)-th Hermite polynomial with highest coefficient 1. Because of (1.3) and (1.4) we may expand \( H(x) \) as

\[ H(x) = \sum_{j=1}^{\infty} c_j H_j(x) \]

(1.5)

with

\[ \sum_{j=1}^{\infty} c_j^2 j! < \infty. \]

(1.6)

We consider the sequence \( H(X_n), n = \ldots, -1, 0, 1, \ldots \) and take the so-called renorm group transformation (see e.g. [1, 27]), i.e. we define the sequences

\[ Y_n^N = \frac{1}{A_N} \sum_{j=1}^{N(n-1)} H(X_j), \quad n = \ldots, -1, 0, 1, \ldots \]

\[ N = 1, 2, \ldots \]

(1.7)

where \( A_N \) is an appropriate positive norming constant. We consider the case \( N \to \infty \), and we are interested in the limit process \( Y_n^* \) if it exists.

In our situation the mixing conditions guaranteeing the central limit theorem with the usual norming factor \( \sqrt{N} \) for sums of weakly dependent random variables (see e.g. [5]) do not hold, and actually both the norming factors and the limit distribution may differ from the usual ones. (Let us remark that by the central limit theorem we mean a slightly stronger statement than it is usually done in the literature. We demand that the sequence defined in (1.7) tend to a sequence of independent normal random variables.)

It was Rosenblatt [6] who first observed these new possibilities (see also [5] 19.5). He proved that in case of \( H(x) = x^2 - 1 \) the limit distribution may be non-Gaussian. The problem was later investigated by Taqqu [8]. He proved that the case of a general \( H(x) \) can be reduced to the case \( H(x) = H_j(x) \), and gave a complete solution for the problem in case \( j = 1, 2 \).

In paper [2] it was proven that any such limit process has to be self-similar.

In the present paper we show that in the case \( c_1 = c_2 = \ldots = c_{k-1} = 0, c_k \neq 0, \alpha < \frac{1}{k} \)

the limit process exists and belongs to a class of self-similar processes which was constructed in [1] by means of multiple Wiener-Itô integrals. (It was called Itô integral in [1].) Our method based on the properties of the Wiener-Itô integrals is different from that of the papers [6] and [8].

Now we formulate