Geometric Convergence of Semi-Markov Transition Probabilities

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1. Introduction

Let \( \{z(t); t \geq 0\} \) be an irreducible semi-Markov process whose transitions occur at instants \( 0 = T_0, T_1, T_2, \ldots \) Suppose \( z(t) \) is continuous to the right and \( z_n = z(T_n), n \geq 0 \). We shall consider a separable version of the process defined by the initial distribution \( \alpha_i = \Pr\{z_0 = i\} \) and the transition probabilities

\[
Q_{ij}(t) = \Pr\{z_n = j, X_n \leq t \mid z_{n-1} = i\},
\]

where \( i, j \) range over a denumerable state space \( E \), and \( X_n = T_n - T_{n-1}, n \geq 1 \).

The occupation time distribution is given by

\[
H_i(t) = \Pr\{X_n \leq t \mid z_{n-1} = i\} = \sum_{j \in E} Q_{ij}(t).
\]

It will be assumed throughout that a transition occurs at \( t = 0 \). Let \( P_{ij}(t) \) be the transition probability from \( i \) to \( j \) in time \( t \), \( G_{ij}(t) (i \neq j) \) the distribution of the first entrance time from \( i \) to \( j \), and \( G_{ii}(t) \) the distribution of the first return time to \( i \) (after leaving \( i \) at the first exit time). Furthermore, let \( kP_{ij}(t) \) denote the transition probability from \( i \) to \( j \) in time \( t \) under the taboo \( k \), and \( kG_{ij}(t) \) the distribution of the first entrance time (or return time, if \( i = j \)) under the taboo \( k \). We agree to impose the taboo only after the first exit from \( i \). Note that \( jP_{ij}(t) = \delta_{ij}\{1 - H_j(t)\} \). \( \eta_j \) and \( \mu_{ij} \) will denote the expectations of \( H_j \) and \( G_{ij} \) respectively, i.e. \( \eta_j = \int_0^\infty t dH_j(t) \) and \( \mu_{ij} = \int_0^\infty t dG_{ij}(t) \).

As Laplace transforms will be used extensively, we shall henceforth adopt the abbreviation \( A^*(s) \) for \( \int_0^\infty e^{-st} dA(t) \).

The notation used here is similar to Pyke and Schaufele's (1964), to whose paper we refer the reader for the definition of a semi-Markov process and a more detailed treatment of the quantities defined above.

Smith (1955) proved that if \( \eta_j < \infty \), then as

\[
t \to \infty, \quad P_{ij}(t) \to \pi_{ij} = G_{ij}(\infty) \eta_j/\mu_{ij}. \quad (\pi_{ij} = 0 \text{ if } \mu_{ij} \text{ is infinite}.)
\]

In section 4 of this paper, we shall establish a necessary and sufficient condition for this convergence to be exponentially fast in the special case when the state space \( E \) is finite. It turns out that when \( E \) is finite, there is geometric convergence

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if there exists $\gamma > 0$ such that $Q^*_{ij}(-\gamma)$ converges and
\[
\frac{Q^*_{ij}(-\gamma + i \delta)}{-\gamma + i \delta} \in L_1(-\infty, \infty)
\]
for all $i, j$ in $E$.

In section 3, a ‘solidarity’ theorem similar to that obtained in Markov chain theory will be proved. A transition $i \to j$ is said to be geometrically ergodic if there exists a positive number $\lambda_{ij}$ such that $P_{ij}(t) - \pi_{ij} = o(e^{-\lambda_{ij}t})$ as $t \to \infty$. Kendall (1959) proved that in an irreducible discrete Markov chain, if for some $k$ the transition $k \to k$ is geometrically ergodic, then all transitions $i \to j$ are geometrically ergodic. This result was improved upon by Vere-Jones (1962) who showed that in an irreducible, discrete, geometrically ergodic Markov chain, a common parameter of convergence, $\lambda$, can be found for all transitions $i \to j$. The continuous time analogue of Vere-Jones’ result was proved by Kingman (April 1963, October 1963). A simple method of proof using Laplace transforms was obtained by Cheong (1966). It is this method which will be used here to establish a similar type of solidarity property for semi-Markov processes. The weakness of this method lies in that it cannot treat cases where instantaneous states exist. We are thus forced here to assume that there are no instantaneous states.

Section 2 contains preliminary results that will be made use of later.

2. Preliminary Results

As with Markov chains, the following first entrance formulas hold:
\[
P_{ij}(t) = kP_{ij}(t) + \int_0^t P_{kj}(t-u) dG_{ik}(u), \quad i, j \in E;
\]
\[
G_{ij}(t) = kG_{ij}(t) + \int_0^t G_{kj}(t-u) d_j G_{ik}(u), \quad i, j, k \in E, \quad j \neq k.
\]

$G_{ij}$ and $P_{ij}$ may also be represented by $Q_{ij}$, thus:
\[
P_{ij}(t) = \delta_{ij} \{1 - H_j(t)\} + \sum_{k \in E} \int_0^t P_{kj}(t-u) dQ_{ik}(u), \quad i, j \in E;
\]
\[
G_{ij}(t) = Q_{ij}(t) + \sum_{k \in E} \int_0^t G_{kj}(t-u) dQ_{ik}(u), \quad i, j \in E.
\]

Considering special cases of (2.1) and (2.2) and taking Laplace transforms, we get the following identities:
\[
P^{*}_{aa}(s) = \frac{1 - H^*_j(s)}{1 - G^*_a(s)},
\]
\[
G^{*}_{aa}(s) = kG^{*}_{aa}(s) + aG_{ak}^{*}(s) G_{ka}^{*}(s),
\]
\[
G^{*}_{ka}(s) = kG^{*}_{ka}(s) + aG_{kk}^{*}(s) G_{ka}^{*}(s),
\]
\[
G_{ak}^{*}(s) = \frac{aG_{ak}(s)}{1 - G^*_a(s)},
\]
\[
G^{*}_{kk}(s) = aG^{*}_{kk}(s) + kG^{*}_{ka}(s) G_{ak}^{*}(s),
\]