Infinitely Divisible Stochastic Processes

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Summary. It often happens that a stochastic process may be approximated by a sum of a large number of independent components no one of which contributes a significant proportion of the whole. For example the depth of water in a lake with many small rivers flowing into it from distant sources, or the point process of calls entering a telephone exchange (considered as the sum of a number of point processes of calls made by individual subscribers) may approximately fulfil these hypotheses. In the present work we formulate and solve the problem of characterizing stochastic processes all of whose finite-dimensional distributions are infinitely divisible. Although some of our results could be derived from known theorems on probabilities on general algebraic structures, many could not and it seems preferable to take the vector-valued infinitely divisible laws as our starting point. We emphasize that an infinitely divisible process (in our sense) on the real line is not necessarily a decomposable process in the sense of Lévy (cf. § 4) though decomposable processes are particular cases.

In § 1 a representation theorem for non-negative (and possibly infinite) stochastic processes is derived, while an analogous theorem in the real-valued case is to be found in § 2. § 3 is devoted to the statement of a "central limit theorem" and the investigation of some of the properties of infinitely divisible processes. The investigation is continued in § 4 by an examination of processes on the real line giving, for example, a representation theorem under weak conditions for infinitely divisible processes which are a.s. sample continuous. Finally in § 5 a study is made of infinitely divisible point processes and random measures.

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1. The Representation Theorem (Non-Negative Case)

One of the most important problems in the theory of probability has been the central limit problem in which the class of distributions of sums of small independent components is investigated. The solutions in the real and vector-valued cases are well known, and in this section and the next the problem is investigated for various types of stochastic process, resulting in a solution for almost any particular stochastic process encountered in probability theory.

First let us consider a.s. non-negative processes taking values in $\mathbb{R}_+ = [0, \infty]$. As a preliminary we recall the infinitely divisible laws in $\mathbb{R}_+^m$. A vector $X$ is said to be infinitely divisible if there exists a set $X_{r,k}$ ($r = 1, 2, \ldots; k = 1, 2, \ldots, r$) of random vectors such that for any fixed $r$ the $X_{r,k}$ are independently and identically distributed and

$$X \sim \sum_{k=1}^{r} X_{r,k},$$

i.e. $X$ has the same distribution as the sum on the right hand side. Let $h(s)$ (called the $h$-function of $X$) be the negative of the logarithm of the Laplace transform of the distribution function of $X$; thus

$$h(s) = -\log E \exp(-s \cdot X) \quad (s \in \mathbb{R}_+^m, \text{ where } \mathbb{R}_+ = (0, \infty)).$$
(Notice that $h(.)$ is finite provided $|X|$ is not a.s. infinite.) Then the $h$-functions of non-negative infinitely divisible random vectors are given by the following theorem:

**Theorem.** A random vector $X$ is infinitely divisible if and only if its $h$-function can be put in the form

$$h(s) = s.a + \int (1 - e^{-\nu u}) \lambda(du)$$

where $a$ is a constant and $\lambda(.)$ a measure on $\mathbb{R}_+^m$ with $\lambda\{0\} = 0$. Unless $|X|$ is a.s. infinite this form is unique and $h(s)$ is finite for all $s$; on the other hand if $|X|$ is a.s. infinite neither is the form unique nor is $h(s)$ finite, at least if all the components of $s$ are positive. Moreover, $|X|$ is infinite with positive probability only if

$$\lambda\{\sum u_i = \infty\} > 0.$$  

(The above theorem can be deduced on the lines followed by Kendall [2] in the one-dimensional case; cf. Gnedenko and Kolmogorov [1], Lévy [7]). We note that it follows from elementary considerations that any non-negative random vector $X$ for which $P\{0 < X_i < \infty\} = 0$ for each $i$ is necessarily infinitely divisible.

Let $T$ be a parameter space, $R(T)$ the vector space sometimes denoted $R_T^m$ of all real-valued functions defined on $T$; $R^m(T)$ the non-negative cone of $R(T)$, and $k^m(T)$ the extension of $R^m(T)$ to include functions taking the value $\infty$. Each of these can be given the structure of a measurable space by associating with it the $\sigma$-ring $\mathcal{B}$ generated by the class of sets of the form

$$\{x; x \in R(T) \ (or \ R^m(T) \ or \ k^m(T)), \ x(t) \in B\}$$

where $t \in T$ and $B$ is a Borel subset of the real line. Let $S(T) = R(T) \setminus \{x; x = 0\}$, $S^m(T) = R^m(T) \setminus \{x; x = 0\}$, $S^m(T) = k^m(T) \setminus \{x; x = 0\}$. Again, each of these becomes a measurable space on associating with it the $\sigma$-ring $\mathcal{F}$ obtained by replacing $R$ by $S$ in the definition of $\mathcal{B}$. (Strictly $\mathcal{B}$, and also $\mathcal{F}$ are ambiguously defined but no real ambiguity will result from our referring always to the $\sigma$-rings appropriate for the case under consideration as $\mathcal{B}$ and $\mathcal{F}$; the importance of $\{x; x = 0\}$ will emerge later.) Notice that $\mathcal{B}$ is always a $\sigma$-field, but $\mathcal{F}$ is a $\sigma$-field (with $S(T), S^m(T)$ or $S^m(T)$ as unity) if and only if $T$ is countable. If now $(\Omega, \mathcal{F}, P)$ is a probability space, a stochastic process $X$ is defined as a measurable function from $(\Omega, \mathcal{F}, P)$ to $(R(T), \mathcal{B})$, and a non-negative (and possibly infinite) stochastic process as a measurable function with range $R^+(T)\cup k^+(T)$. Stochastic processes $X_k$ are said to be independent if the inverse images under $X_k$ of $\mathcal{B}$ are independent $\sigma$-rings, i.e. events selected arbitrarily, one from each class, are independent. Clearly, if $X_1$ and $X_2$ are stochastic processes defined on the same probability space, the vector space structure allows us to define a stochastic process $aX_1 + + \beta X_2$, where $a, \beta \in R$. (For all the above, cf. Lévy [8].)

A (non-negative) stochastic process $X$ is said to be infinitely divisible if there exists a set $X_r, k (r = 1, 2, \ldots; k = 1, 2, \ldots, r)$ of (non-negative) stochastic processes such that for any fixed $r$ the $X_r, k$ are independently and identically distributed and $X$ has the same finite-dimensional distributions as $\sum_{k=1}^{r} X_r, k$. The purpose of