The Kolmogorov Equation for a Plane Barrier Problem

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Summary. The probability density $p$ of a plane Brownian motion stopped by a two-sided constant barrier is shown to be a solution of a Kolmogorov forward equation of the form $\mathcal{L}^*p = 0$. The operator $\mathcal{L}^*$ is the product of two second order differential operators, each of them corresponding to a related one-dimensional Brownian motion.

1. The Expectation of Plane Stochastic Differentials

A plane Brownian motion or Wiener measure on the square $S = (0, 1] \times (0, 1]$ is a probability measure on the space of continuous functions $\beta : S \to R$ such that the variables $\{\beta(x, y) | (x, y) \in S\}$ are a Gaussian family and $E(\beta(x, y)) = 0$, $\text{Cov}(\beta(x, y), \beta(x', y')) = (x \wedge x')(y \wedge y')$ for all $(x, y), (x', y') \in S$ (for a construction of such process we refer to [1]).

Let $\mathcal{A}$ be a family of $\sigma$-fields defined on $S$, increasing in each of the variables $x, y$ and such that for each $(x, y) \in S$, $\mathcal{A}(x, y)$ contains the $\sigma$-field of $\{\beta(x', y') | 0 \leq x' \leq x, 0 \leq y' \leq y\}$ and is independent of the $\sigma$-field generated by $\{\beta(x', y') - \beta(x' \wedge x, y' \wedge y) | (x', y') \in S\}$.

A random functional $e$ defined on $S$ is said to be nonanticipating when it is measurable over the product of the Borel $\sigma$-field in $S$ and $\mathcal{A}(1, 1)$, and, for each $(x, y) \in S$, $e(x, y)$ is $\mathcal{A}(x, y)$-measurable.

Let $\mathcal{P}_n = \{R_{i,j}^{(n)} | i, j = 1, 2, \ldots, 2^n\}$ denote the partition of $S$ into the $2^n$ equal squares

$$R_{i,j}^{(n)} = \{(x, y) | t_{i-1}^{(n)} \leq x < t_i^{(n)}, t_{j-1}^{(n)} \leq y < t_j^{(n)}\} \quad (i, j = 1, 2, \ldots, 2^n)$$

where

$$t_i^{(n)} = i \cdot 2^{-n} \quad (i = 0, 1, \ldots, 2^n).$$

A functional $e$ is said to be simple when there exists a positive integer $n$ such that $e$ remains constant (as a function of $(x, y)$) on each square $R_{i,j}^{(n)}$ of $\mathcal{P}_n$. For
simple and nonanticipating $e$, its double stochastic integral with respect to $\beta$ is defined in the customary way as

$$\int_{S(x,y)} e \, d\beta = \sum_{ij} e(i_{l-1}, j_{l-1}) \beta(R_{ij} \cap S(x,y)),$$

(1)

where

$$S(x,y) = (0, x] \times (0, y],$$

and for any rectangle $R=(x', x''] \times (y', y'']$, $\beta(R)$ means the double increment

$$\beta(R) = \square R = \beta(x'', y'') - \beta(x', y'') - \beta(x', y') + \beta(x', y').$$

(Notice that the previous formula defines the symbol $\square$.)

The stochastic integral $\int_{S(x,y)} e \, d\beta$ of a nonanticipating $e$ such that $P\{\int_S e^2 \, dx \, dy < \infty\} = 1$ on the rectangle $S(x,y) = (0, x] \times (0, y]$ can be defined and constructed following the same steps as in [2] (pp. 21–24) with the obvious modifications, as a uniform limit of stochastic integrals of simple functionals.

The well known Itô's formula for stochastic differentiation, expresses that, if $u(t, z)$ has continuous partial derivatives $\partial u/\partial t$, $\partial u/\partial z$, $\partial^2 u/\partial z^2$, and the stochastic integral with respect to a (one-dimensional) Brownian motion $b$

$$c(t) = c(0) + \int_0^t e(\tau) \, dB(\tau)$$

is substituted for $z$, the composite function $u(t, c(t))$ has stochastic differential

$$du(t, c(t)) = \left( \frac{\partial}{\partial t} + \frac{1}{2} e^2(t) \frac{\partial^2}{\partial z^2} \right) u \, dt + \frac{\partial}{\partial z} u \, dc(t),$$

(2)

and hence

$$Edu(t, c(t)) = E \left( \frac{\partial}{\partial t} + \frac{1}{2} e^2(t) \frac{\partial^2}{\partial z^2} \right) u \, dt.$$  

(3)

That formula plays an important role in the derivation of Kolmogorov equations for $c$, but it suffices for that purpose to use the weaker version (3) instead of (2), as it is easily verified (see, for instance, [21, 3.5].)

The generalization of (2) to plane integrals is not plain [3]. Nevertheless, (3) can be easily generalized, as the following lemma shows.

**Lemma 1.** If $e$ is a vector-valued nonanticipating functional $e=(e_1, e_2, \ldots, e_k)$ defining $\gamma(x,y)=\int_{S(x,y)} e \, d\beta$, and $u(x,y,z)$ is any bounded function in $C^\infty(S \times R^k)$ ($z=(z_1, z_2, \ldots, z_k)$), then

$$E \square_z u(x,y, \gamma(x,y)) = E \int_S L(e) u \, dx \, dy$$

(4)