On the Entropy of a Product Endomorphism in Infinite Measure Spaces

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Summary. In this note, a relationship is established between the entropy, defined by Krengel for an endomorphism of a σ-finite measure space, and the notion of a spreading partition. This relationship is used to answer in the "quasi-finite" case a question raised by Krengel concerning the entropy of the product endomorphism on the direct product of a finite and σ-finite measure space.

Let \((\Omega, \mathcal{F}, \mu)\) be a σ-finite measure space on which is defined a conservative endomorphism \(\tau\). Let \(\mathcal{F} = \{F_i\}\) be a countable partition of \(F\), \(\mu(F) < \infty\). The entropy of \(\mathcal{F}\), \(H(\mathcal{F}) = -\sum \mu(F_i) \log \mu(F_i)\). For any set \(F\), the first return partition \(\mathcal{E}_F\) of \(F\) consists of the sets \(E_0 = \emptyset\) and for \(n = 1, 2, \ldots\)

\[ E_n = (F \cap \tau^{-n} F) - \bigcup_{i=0}^{n-1} E_i. \]

For conservative endomorphisms, \(F = \bigcup_{i=1}^\infty E_i\) follows from the recurrence theorem [6]. The endomorphism \(\tau\) induces an endomorphism \(\tau_F\) on \(F\) defined by

\[ \tau_F(\omega) = \tau^m(\omega) \quad \text{for} \quad \omega \in E_n. \]

Krengel [4], defined the entropy \(h(\tau)\) of \(\tau\) by

\[ h(\tau) = \sup_{\mu(F) < \infty} h(\tau_F) \]

where \(h(\tau_F)\) is the entropy of the induced transformation \(\tau_F\) on the finite measure space \((F, F \cap \mathcal{F}, \mu)\).

We now relate the Krengel entropy to the notion of a spreading partition introduced in [3]. Given any two partitions \(\mathcal{F}\) of \(F\) and \(\mathcal{G}\) of \(G\), the refinement of \(\mathcal{F}\) and \(\mathcal{G}\) is the partition \(\mathcal{F} \vee \mathcal{G}\) of \(F \times G\) consisting of sets of the form:

\[ F_i \cap G_j, \quad F^c \cap G_j, \quad F_i \cap G^c. \]

This method of refinement extends to any finite number of partitions and is easily seen to be commutative and associative.

Given a partition \(\mathcal{F}\) of \(F, \mu(F) < \infty\), and a σ-field \(\mathcal{G}\) of subsets of \(G\) on which \(\mu\) is σ-finite, the conditional information of \(\mathcal{F}\) given \(\mathcal{G}\), \(J(\mathcal{F}|\mathcal{G})\) is:

\[ J(\mathcal{F}|\mathcal{G}) = \begin{cases} 
(i) \ -\sum 1_{F_i} \log \mu(F_i|\mathcal{G}) & \text{on } F \cap G \\
(ii) \ -\log \mu(F^c|\mathcal{G}) & \text{on } F^c \cap G \\
(iii) \ -\sum 1_{F_i} \log \mu(F_i \cap G^c) & \text{on } F \cap G^c \\
(iv) \ 0 & \text{on } F^c \cap G^c.
\end{cases} \]
This definition coincides with the usual definition on $G$ but not on $G'$. In the sequel, we shall use the notation:

$$\mathcal{F}_i^j = \bigvee_{k=i}^j \tau^{-k} \mathcal{F}.$$  

The partition $\mathcal{F}_i^n$ "spreads over the entire space $\Omega$". We now relate Krengel's entropy to the conditional information of a spreading partition.

**Theorem 1.** Let $F$ be a set, $\mu(F) < \infty$, for which $H(\mathcal{F}_F) < \infty$. Then

$$h(F) = \sup \left\{ J(\mathcal{F} | F \cap \mathcal{F}_F^\infty) \mu : \tau^{-i} F \subset \mathcal{F}_F^i \right\}$$

where supremum is taken over all partitions of $F$ for which $H(\mathcal{F}) < \infty$.

**Proof.** Let $\mathcal{F}$ be any partition of $F$ for which $H(\mathcal{F}) < \infty$ and $\mathcal{F}_F \subset \mathcal{F}$. The relation

$$J(\mathcal{F} | F \cap \mathcal{F}_F^\infty) = J(Y | F_n, Y)$$

has been used by Scheller ([5], p. 44) and by Krengel ([4], p. 175). The reverse inclusion follows essentially from Lemma 5 of [1] where it is assumed that $\tau$ is an automorphism; however, the same proof may be used for endomorphisms. Therefore

$$J(\mathcal{F} | F \cap \mathcal{F}_F^\infty) = J(Y | F_n, Y)$$

The expression to the left in (3) is the usual conditional entropy of the partition $\mathcal{F}$ of the finite measure space $(F, F \cap \mathcal{A}, \mu)$. Since $H(\mathcal{F}_F) < \infty$, it suffices to take supremum over all partitions of $F$ for which $\mathcal{F}_F \subset \mathcal{F}$ and $H(\mathcal{F}) < \infty$.

We now compute the entropy of a direct product of two endomorphisms $\tau_X$ and $\tau_Y$ of the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ when $h(\tau_Y) = 0$ and $\lambda(Y) < \infty$. It is easily seen that $\tau_X \times \tau_Y$ is conservative. A sweep out set $F$ is one for which

$$\bigcup_{i=0}^\infty \tau^{-i} F.$$  

**Theorem 2.** Let $\tau_X \times \tau_Y$ be the product endomorphism on $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \lambda)$. Let $F \subset X$ be a sweep out set whose first return partition has finite entropy; let $\lambda(Y) < \infty$ and $h(\tau_Y) = 0$. Then

$$h(\tau_X \times \tau_Y) = \lambda(Y) h(\tau_X).$$

**Proof.** Let $E_n, n=1, 2, \ldots$ be the first return partition of $F$. Then, the first return partition of $F \times Y$ consists of the sets: $E_n \times Y, n=1, 2, \ldots$. Moreover, the set $F \times Y$ is a sweep out set and by Theorem 3.1 of [4], $h(\tau_X \times \tau_Y) = h(F \times Y)$, where $\tau_{F \times Y}$ is the transformation on $F \times Y$ induced by $\tau_X \times \tau_Y$. To compute the entropy of $\tau_{F \times Y}$, it suffices to consider partitions of $F \times Y$ consisting of rectangles (see [2], p. 277 ff.). Moreover, it suffices to consider partitions which contain the first return partitions of $F \times Y$. Therefore,

$$h(\tau_X \times \tau_Y) = \sup_{\mathcal{F} \times \mathcal{G}} \left\{ J(\mathcal{F} \times \mathcal{G} | F \times Y \cap (\mathcal{F} \times \mathcal{G})^\infty) \mu \times \lambda \right\}.$$