Geometric Ergodicity in a Class of Denumerable Markov Chains*

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Abstract. We study the question of geometric ergodicity in a class of Markov chains on the state space of non-negative integers for which, apart from a finite number of boundary rows and columns, the elements \( p_{jk} \) of the one-step transition matrix are of the form \( c_k \delta_{j} \) where \( \{c_k\} \) is a probability distribution on the set of integers. Such a process may be described as a general random walk on the non-negative integers with boundary conditions affecting transition probabilities into and out of a finite set of boundary states. The imbedded Markov chains of several non-Markovian queueing processes are special cases of this form. It is shown that there is an intimate connection between geometric ergodicity and geometric bounds on one of the tails of the distribution \( \{c_k\} \).

1. Introduction

Consider a homogeneous, irreducible, aperiodic Markov chain with a countable number of states identified by the non-negative integers. We denote the transition probability matrix by \( P = (p_{jk}) \), where \( p_{jk} \) (\( j, k = 0, 1, 2, \ldots \)) is the one-step transition probability from state \( j \) to state \( k \). Let \( P^n = \{p_{jk}^{(n)}\} \) be the matrix of \( n \)-fold transition probabilities. It is well known (see, eg., Chung [3]) that for each \( j, k \) the limit

\[
\lim_{n \to \infty} p_{jk}^{(n)} = \pi_k
\]

exists; this limit is positive for all pairs \( j \) and \( k \) if the chain is ergodic and zero if the chain is null-recurrent or transient. The chain is said to be geometrically ergodic (Kendall [9]) if for each pair of states \( j, k \) the rate of approach of \( p_{jk}^{(n)} \) to its limit is geometrically fast. More precisely, the chain is geometrically ergodic when numbers \( M_{jk} \) and \( \varrho_{jk} \) exist such that

\[
0 \leq M_{jk} < \infty, \quad 0 \leq \varrho_{jk} < 1,
\]

\[
|p_{jk}^{(n)} - \pi_k| \leq M_{jk} \varrho_{jk}^n \quad (n = 0, 1, 2, \ldots),
\]

for all pairs of states \( j \) and \( k \). Kendall showed that the property of geometric ergodicity is a class property of an irreducible set of states in the sense that the geometric rate of approach for one state implies that for all pairs of states. More precisely again, an irreducible aperiodic Markov chain will be geometrically ergodic if and only if

\[
|p_{00}^{(n)} - \pi_0| < M \varrho^n
\]

for some finite non-negative \( M \) and some \( \varrho \) satisfying \( 0 \leq \varrho < 1 \). State 0 is here meant to represent any given state, the choice of 0 being a matter of labelling only.

Vere-Jones [15] went further and showed that the rate parameters \( \varrho_{jk} \) in (1.1) may all be replaced by a single parameter \( \varrho \) (\( 0 \leq \varrho < 1 \)) uniform for all pairs of states.

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Kendall [10] and Vere-Jones [16] examine the question of geometric ergodicity for some particular Markov chains, namely the imbedded Markov chains of certain queueing process. For example, Kendall considers the queing system $M/G/1$. In this system there is a single server; customers, arriving in a Poisson process of rate $\beta$, are served in order of arrival. The service times of successive customers are independent, identically distributed random variables with distribution function $S(\cdot)$ which is assumed to have a finite non-zero mean, conveniently taken to be the unit of time. It is further assumed that $S(0+) = 0$. The Poisson rate $\beta$ is in fact the traffic intensity, i.e., the ratio of the mean service time to the mean inter-arrival time. If we consider the number of customers present (waiting or being served) immediately after each successive departure then this number forms a Markov chain for which the one-step transition matrix has the form

$$P = \begin{bmatrix}
  a_0 & a_1 & a_2 & a_3 & \ldots \\
  a_0 & a_1 & a_2 & a_3 & \ldots \\
  0 & a_0 & a_1 & a_2 & \ldots \\
  0 & 0 & a_0 & a_1 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad (1.2)$$

where

$$a_n = \int_0^{\infty} e^{-\beta \lambda} \left\{ \frac{(\beta \lambda)^n}{n!} dS(\lambda) \right\} \quad (n = 0, 1, 2, \ldots) .$$

(For a full derivation of this result see Kendall [7]).

All the imbedded Markov chains considered by Kendall and Vere-Jones have a property in common. They may all be described as being of the random walk type, by which we mean that the transition probabilities $p_{jk}$ are, apart from a finite number of boundary rows and columns, functions of $k - j$ only. That is, the $p_{jk}$ are, apart from a finite number of boundary rows and columns, constant along any one diagonal of $P$. The matrix (1.2) in the above example clearly has this property. In general these Markov chains are random walks on the non-negative integers subject to certain boundary conditions.

The aim of the present paper is to consider the geometric ergodicity of a random walk on the non-negative integers whose increments are governed by a general distribution $\{c_j; j = 0, \pm 1, \pm 2, \ldots\}$. The walk is subject to boundary conditions affecting one-step transition probabilities into and out of the finite set of boundary states $(0, 1, \ldots, z)$. The one-step transition matrix is of the form

$$P = \begin{bmatrix}
  p_{00} & p_{01} & \cdots & p_{0z} & p_{0, z+1} & p_{0, z+2} & \cdots \\
  p_{10} & p_{11} & \cdots & p_{1z} & p_{1, z+1} & p_{1, z+2} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
  p_{z, 0} & p_{z, 1} & \cdots & p_{z, z} & p_{z, z+1} & p_{z, z+2} & \cdots \\
  p_{z+1, 0} & p_{z+1, 1} & \cdots & c_0 & c_1 & c_2 & \cdots \\
  p_{z+2, 0} & p_{z+2, 1} & \cdots & c_{-1} & c_0 & c_1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \quad (1.3)$$

Here $p_{jk} = c_{k-j}$ for $j > z$ and $k > z$, while otherwise the $p_{jk}$ are arbitrary, but given, and are subject of course to the conditions

$$\sum_{k=0}^{\infty} p_{jk} = 1 \quad (j = 0, 1, \ldots, z), \quad (1.4)$$

$$p_{jo} + p_{j1} + \cdots + p_{jz} = \sum_{i=-\infty}^{-j} c_i \quad (j > z). \quad (1.5)$$