Remarks on a Markov Chain Example of Kolmogorov

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Summary. It is shown that the stochastic transition matrix $P(t)$, in Kolmogorov’s example of a process with an instantaneous state, is uniquely determined by the derivative matrix $Q = P'(0)$, and the most general such substochastic $P(t)$ is also found. The example is used to show that, if 0 is an instantaneous state, then $1 - p_{00}(t)$ can tend to 0 arbitrarily slowly and on the other hand $(1 - p_{00}(t))/t$ can tend to $+\infty$ arbitrarily slowly.

1. We denote by $P(t) = (p_{ij}(t))$ a standard transition matrix, possibly substochastic, where $i$ and $j$ range over the non-negative integers, and write $q_{ij} = p_{ij}(0)$, $Q = (q_{ij})$.

The first example ("K 1") for which $Q$ contains $-\infty$ on the diagonal, so that the Markov chain with transitions $P(t)$ has an instantaneous state, was given by Kolmogorov [4] and further discussed by Kendall and Reuter [3] and Chung [1]. In K 1, the matrix $Q$ has the form

$$Q = \begin{pmatrix}
\infty & b_1 & b_2 & b_3 & \ldots \\
-a_1 & 0 & 0 & 0 & \ldots \\
a_2 & 0 & -a_2 & 0 & \ldots \\
a_3 & 0 & 0 & -a_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix},$$

(1)

where $a_n > 0$, $b_n > 0$, $\sum b_n = \infty$, $\sum b_n |a_n| < \infty$ (this mildly generalises K 1, which had all $b_n = 1$). Professor D. A. Freedman recently asked whether $Q$ determines a unique stochastic $P(t)$ with $P'(0) = Q$, and I will show that it does and also determine all substochastic $P(t)$ with $P'(0) = Q$. The only non-trivial part of the argument is a simple analytical proof that the forward equations

$$p'_{ij}(t) = \sum_k p_{ik}(t) q_{kj}$$

(2)

must hold when $j \geq 1$; Professor K. L. Chung has pointed out to me that there is a simple (but less elementary) probabilistic proof, using the observation that sample functions of a process with $Q$ as in (1) cannot have pseudo-jumps from $\infty$ to $j \geq 1$ and then using Theorem 16.3 of his book [1].

We begin the proof of (2) by observing that the backward equations

$$p'_{ij}(t) = \sum_k q_{ik} p_{kj}(t)$$

(3)

must hold for $i \geq 1$ because $\sum_k q_{ik} = 0$; this follows from [1; Theorem II 17.2] or by an easy direct argument. Writing out (3) for our $Q$, we have

$$p'_{ij}(t) = a_i p_{0j}(t) - a_i p_{ij}(t),$$

$$p_{ij} = \delta_{ij} \varphi_t + a_i p_{0j} \varphi_t,$$

(4)
where * denotes convolution and $\varphi_i(t) = e^{-at_i}$. To get (2) for $j \geq 1$, let $s \to 0$ in

$$s^{-1}(p_{ij}(t+s) - p_{ij}(t)) = \sum_k p_{ik}(t) \frac{p_{kj}(s)}{s} - \delta_{kj}$$

$$= p_{i0}(t) \frac{p_{0j}(s)}{s} + \sum_{k \neq 0, j} p_{ik}(t) \frac{p_{kj}(s)}{s} + \sum_k p_{ik}(t) \frac{p_{kj}(s)}{s};$$

it will be enough to show that the last term tends to zero as $s \to 0$. To estimate $p_{kj}(s)$ use (4), i.e.

$$p_{kj} = p_{0j} * a_k \varphi_k,$$

and the fact that $p_{0j} \leq C t$ because $p'_{0j}(0) < \infty$. Thus

$$p_{kj}(s) \leq C \int_0^s u a_k e^{-a_k(s-u)} du = C \int_0^s (1 - e^{-a_ku}) dv,$$

and so

$$\sum_{k \neq 0, j} p_{ik}(t) \frac{p_{kj}(s)}{s} \leq C \sum_{k \neq 0, j} p_{ik}(t) \frac{1}{s} \int_0^s (1 - e^{-a_ku}) dv,$$

say. But $1 \geq \psi_k(s) \downarrow 0$ as $s \downarrow 0$, so that

$$\sum_{k \neq 0, j} p_{ik}(t) \frac{p_{kj}(s)}{s} \to 0$$

as required and (2) is established. Solving (2) we get

$$p_{ij} = \delta_{ij} \varphi_j + b_j p_{i0} \varphi_j \quad (j \geq 1). \quad (5)$$

We can now use (4) and (5) to express all $p_{ij}$ in terms of $p_{00}$, which we denote by $\varphi$. We leave it to the reader to write down the general formula, but record the particular case

$$p_{0j} = b_j \varphi \star \varphi_j \quad (j \geq 1). \quad (6)$$

2. It remains to determine the general form of $\varphi = p_{00}$. If we require $P(t)$ to be stochastic, then summation over $j$ in (6) leads to

$$\varphi(t) + \int_0^t k(u) \varphi(t-u) du = 1,$$

where

$$k(u) = \sum_{n \geq 1} b_n e^{-a_n u}. \quad (8)$$

This determines the bounded continuous function $\varphi$ uniquely; the other $p_{ij}$ can be expressed in terms of $\varphi$, so that we have shown that there is at most one stochastic $P(t)$ with $P'(0) = Q$ as in (1). Verification that this $P(t)$ is a standard transition matrix with $P'(0) = Q$ can be provided by trivial extensions of the arguments in [4], [3], or [1]. There is therefore exactly one stochastic $P(t)$ with $P'(0) = Q$.

Now drop the requirement that $P(t)$ be stochastic, so that we can no longer use (7) to determine $\varphi$. Instead, use the Laplace transform (resolvent) $R(\lambda)$ of $P(t)$